EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST EXAMINATION IN SCIENCE - 2016/2017
FIRST SEMESTER (Aug./Sept., 2018)
AM 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I
(REPEAT)

Answer all questions | Time : Three hours

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence show that

$$(\underline{a} \wedge \underline{b}) \cdot [(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})] = [\underline{a} \cdot (\underline{b} \wedge \underline{c})]^2.$$ [40 marks]

(b) Let the vector $\underline{x}$ be given by the equation $\lambda \underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}, \underline{b}$ are constant vectors and λ is a non-zero scalar. Show that $\underline{x}$ satisfies the equation

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

Hence find $\underline{x}$ in terms of $\underline{a}, \underline{b}$ and λ. [25 marks]

(c) Find the vector $\underline{x}$ and the scalar λ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda \underline{a}, \qquad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$. [35 marks]

2. (a) Define the following terms:

- the gradient of a scalar field;
- the curl of a vector field.

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(b) Prove that if ϕ is a scalar field and $\underline{A}$ is a vector field then

$$\text{curl}(\phi\underline{A}) = \phi\ \text{curl}\underline{A} + \text{grad}\phi \wedge \underline{A}.$$

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(c) Let $\underline{a}$ be a non zero constant vector and let $\underline{r}$ be a position vector of a p that $\underline{a} \cdot \underline{r} \neq 0$, and let n be a constant. If $\phi = (\underline{a} \cdot \underline{r})^n$, then show that ∇ and only if $n = 0$ or $n = 1$.

If $r = |\underline{r}|$, find grad $\left(\frac{\underline{a} \cdot \underline{r}}{r^5}\right)$. Hence show that

$$\text{curl}\left(\frac{\underline{a} \cdot \underline{r}}{r^5}\ \underline{r}\right) = \frac{\underline{a} \wedge \underline{r}}{r^5}.$$

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3. (a) Define the following terms:

- a conservative vector field;
- solenoidal vector field.

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Show that $\underline{A} = (2x-y)\underline{i} + (2yz^2 - x)\underline{j} + (2y^2z - z)\underline{k}$ is conservative but not s

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(b) Let $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,2,0)$ and $C = (1,2,3)$. By consid straight line path OA, AB, BC, find the line integral $\int_\gamma \underline{A} \cdot d\underline{r}$, where $\underline{A} = (2y+3)\underline{i} + xz\underline{j} + (yz - x)\underline{k}$ and γ is a path from O to C. [3

(c) State the *Divergence theorem*, and use it to evaluate $\int\int_S \underline{A} \cdot \underline{n}\ dS$, where $\underline{A} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded b $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

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4. (a) Prove that the radial and transverse component of the acceleration of a particle in terms of the polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

respectively.

[30 marks]

(b) A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modules mg and un-stretched length 'a'. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is half of its initial velocity.

[70 marks]

5. (a) A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

[30 marks]

(b) A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body , where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

[70 marks]

6. (a) State the angular momentum principle. [14.

(b) A particle of mass m moves on a smooth inner surface of a paraboloid of r

$r^2 = 4az$ whose axis is vertical and vertex downwards, the path of the pa

between the horizontal circle at $z = p$ and $z = q$. Show that the angular m

of the particle above the axis is $m\sqrt{8agpq}$ and the velocity is $\sqrt{2g(p+q-$

g is the acceleration due to gravity.

Show also that the reaction between the surface and the particle when the

is at $z = p$ is given by

$$\frac{mg(a+q)}{\sqrt{a(a+p)}}.$$

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