EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST EXAMINATION IN SCIENCE -2008/2009
FIRST SEMESTER (Mar./May, 2010)
MT 103 - VECTOR ALGEBRA AND CLASSICAL MECHANICS I

Answer all Questions — Time: Three hours

1. Define the terms *colinear* vectors and *coplanar* vectors.

   (a) Prove the following:

   i. the diagonals of a parallelogram bisect each other,

   ii. if $\underline{a}$, $\underline{b}$ and $\underline{c}$ are non-coplanar vectors then $x\underline{a} + y\underline{b} + z\underline{c} = \underline{0}$ implies

   $x = y = z = 0.$

   (b) Let $\underline{p}, \underline{q}$ and $\underline{r}$ are three non-null vectors such that $\underline{r} - (\underline{p} \wedge \underline{q}) = \alpha \underline{q}$ and $\underline{p} \cdot \underline{q} = 0$, where α is a scalar. Show that

   $$\underline{p} = \underline{q} \wedge \frac{\underline{r}}{|\underline{q}|^2} \text{ and } \alpha = \frac{\underline{q} \cdot \underline{r}}{|\underline{q}|^2}.$$

   (c) Find an equation of plane determined by the points $P_1(2, -1, 1,)$, $P_2(3, 2, -1)$ and $P_3(-1, 3, 2)$.

2. Define the terms *divergence* and *curl* of the vector field $\underline{F}$.

   (a) If $\underline{F} = \underline{F}(x, y, z)$ is a vector function and $\phi = \phi(x, y, z)$ is a scalar function. Prove that

   $$\text{curl}(\phi \underline{F}) = \phi \text{curl} \underline{F} + \text{grad}\phi \wedge \underline{F}.$$

   Hence show that

   $$\text{curl}(\psi \underline{\nabla} \phi) = -\text{curl}(\phi \underline{\nabla} \psi).$$

(b) Prove the following:

i. if $\underline{a}$ and $\underline{b}$ are irrotational vectors, then $\underline{a} \wedge \underline{b}$ is a solenoidal vector,

ii. if $\underline{r} \wedge d\underline{r} = \underline{0}$, then $\hat{\underline{r}} =$ constant.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $|\underline{r}| = r$. Show that:

i. $\operatorname{div}(r^n \underline{r}) = (n+3)r^n$,

ii. $\operatorname{div}\left(\frac{\underline{r}}{r^3}\right) = 0$,

iii. $\underline{\nabla}^2(r^n) = n(n+3)r^{n-2}\underline{r}$.

3. State the *Stoke's theorem* and *divergence theorem.*

(a) If S is any open surface bounded by a simple closed curve C and $\underline{B}$ is any vector then prove that

$$\oint_C d\underline{r} \wedge \underline{B} = \int\int_S (\underline{n} \wedge \underline{\nabla}) \wedge \underline{B}\, ds.$$

(b) Verify the *Stoke's theorem* for $\underline{F} = (x^2 + y - 4)\underline{i} + 3xy\underline{j} + (2xz + z^2)\underline{k}$, S is an upper half of the sphere $x^2 + y^2 + z^2 = 16$ and C is it's boundary.

(c) Evaluate $\int\int_S [(x^3 - yz)\underline{i} - 2x^2y\underline{j} + 2\underline{k}]\cdot\underline{n}ds$ by using divergence theorem,where S denotes the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$.

4. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Write the velocity and acceleration components of the particle in intrinsic coordinate. Using these, show that the components of acceleration along the tangent and perpendicular to it are given by $v\frac{dv}{ds}$ and $v^2\frac{d\psi}{ds}$ respectively.

A smooth wire in the form of an arc of a cycloid $s = 4a\sin\psi$, is fixed in a vertical plane with the vertex downwards and the tangent at the vertex is horizontal. A small bead of mass m is threaded on the wire and is projected from the vertex with speed $\sqrt{8ag}$. If the resistance of the medium in which the motion take place is $mv^2/8a$, when the speed is v then show that the bead comes to instantaneous rest at a cusp ($\psi = \pi/2$) and returns to the starting point with speed $\sqrt{8ag(1 - 2e^{-1})}$.

5. With usual notations, discuss the motion of a particle on a smooth surface of revolution $z = f(r)$ whose axis is verticle.

   Deduce that the motion is possible if $\frac{1}{2g}\left(2E - \frac{\lambda^2}{r^2}\right) \geq f(r)$, where λ is a constant.

   A particle of mass m moves on a smooth inner surface of a paraboloid of revolution $r^2 = 4az$ whose axis is vertical and vertex downwards, the path of the particle lies between the horizontal circle at $z = p$ and $z = q$. Show that the angular momentum of the particle above the axis is $m\sqrt{8agpq}$ and the velocity is $\sqrt{2g(p+q-z)}$.

   Show also that the reaction between the surface and the particle when the particle at $z = p$ is

$$\frac{mg(a+q)}{a(p+q)^{1/2}}.$$

6. With usual notations, obtain the equation of motion of a rocket of varying mass in the form

$$\underline{F}(t) = m(t)\frac{d\underline{v}}{dt} + \underline{v_0}\frac{dm(t)}{dt}.$$

   A rocket with initial mass M is fired upwards. Matter is ejected with relative velocity u at a constant rate kM, where k is a constant. M' is the mass of the rocket without fuel. Neglecting air resistance and variation in gravitational attraction, show that the greatest velocity is given by

$$u\ln\frac{M}{M'} - \frac{g}{k}\left(1 - \frac{M'}{M}\right)$$

   and the greatest height reached is,

$$\frac{u^2}{2g}\left(\ln\frac{M}{M'}\right)^2 + \frac{u}{k}\left(1 - \frac{M}{M'} - \ln\frac{M}{M'}\right).$$