# EASTERN UNIVERSITY, SRI LANKA

## DEPARTMENT OF MATHEMATICS

## EXTERNAL DEGREE EXAMINATION IN SCIENCE 2010/2011

## FIRST YEAR FIRST SEMESTER (Apr./May, 2017)

## EXTMT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

## (SPECIAL REPEAT)

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Answer all questions | Time : Three hours

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}.$$

(b) Let the vector $\underline{x}$ be given by the equation $\lambda \underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}, \underline{b}$ are constant vectors and $\lambda$ is a non-zero scalar. Show that $\underline{x}$ satisfies the equation

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

Hence find $\underline{x}$ in terms of $\underline{a}, \underline{b}$ and $\lambda$.

(c) Find the vector $\underline{x}$ and the scalar $\lambda$ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda \underline{a}, \qquad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$.

2. Define the terms *divergence* and *curl* of a vector field $\underline{F}$.

(a) Let $\underline{F} = \underline{F}(x, y, z)$ be a vector function and $\phi = \phi(x, y, z)$ be a scalar function. Prove that

$$\text{div}(\phi\underline{F}) = \phi\text{div}\underline{F} + \text{grad}\phi \cdot \underline{F}.$$

Hence show that

$$\text{div}(r^n\underline{r}) = (n+3)\ r^n.$$

(b) Define the term *solenoidal vector.*

Show that the vector

$$\underline{A} = (x+3y)\underline{i} + (y-3z)\underline{j} + (x-2z)\underline{k}$$

is solenoidal.

(c) Define the term *irrotational vector.*

Determine the constants $a, b$ and $c$ so that the vector

$$\underline{F} = (x+2y+az)\underline{i} + (bx-3y-z)\underline{j} + (4x+cy+2z)\underline{k}$$

is irrotational.

3. State the *Stokes'* theorem.

(a) Verify the Stokes' theorem for a vector $\underline{A} = (2x-y)\ \underline{i} - yz^2\ \underline{j} - \ y^2z\ \underline{k}$, where S the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and $C$ is its boundary.

(b) Evaluate $\iint_V \phi dV$, where $\phi = 45x^2y$ and $V$ is the closed region bounded by the pl $4x+2y+z=8,\ x=0,\ y=0,\ z=0.$ .

4. A particle of mass $m$ rests on a smooth horizontal table attached through a fixed point the table by a light elastic string of modules $mg$ and unstretched length $'a'$. Initially string is just taut and the particle is projected along the table in a direction perpendic to the line of the string with velocity $\sqrt{\dfrac{4ag}{3}}$. Prove that if $r$ is the distance of the part from the fixed point at time $t$ then

$$\frac{d^2\underline{r}}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial velocity.

5. A particle moves in a plane with the velocity $v$ and the tangent to the path of the particle makes an angle $\psi$ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity $v_0$. The parachute exerts a drag opposing motion which is $k$ times the weight of the body , where $k$ is a constant. Neglecting the air resistance to the motion of the body, prove that if $v$ is the velocity of the body when its path is inclined an angle $\psi$ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. State the angular momentum principle for motion of a particle.

A right circular cone with a semi vertical angle $\alpha$ is fixed with its axis vertical and vertex downwards. A particle of mass $m$ is held at the point $A$ on the smooth inner surface of the cone at a distance $'a'$ from the axis of revolution. The particle is projected perpendicular to $OA$ with velocity $'u'$, where $O$ is the vertex of the cone. Show that the particle rises above the level of $A$ if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$