EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SECOND EXAMINATION IN SCIENCE -2010/2011
FIRST SEMESTER (APRIL, 2013)
PM 201 - VECTOR SPACES AND MATRICES
(PROPER & REPEAT)

Answer all Questions **Time: Three hours**

1. (a) Define what is meant by

   i. a vector space,

   ii. a subspace of a vector space. [10 marks]

   (b) Let $M_{m\times n}$ be the set of all real $m \times n$ matrices. For any two matrices $A = (a_{ij}),\ B = (b_{ij}) \in M_{m\times n}$, and for any $\lambda \in \mathbb{R}$, define an addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$(a_{ij}) \oplus (b_{ij}) = (a_{ij} + b_{ij}),$$
$$\lambda \odot (a_{ij}) = (\lambda a_{ij}).$$

   Prove that $(M_{m\times n}, \oplus, \odot)$ is a vector space over the field $\mathbb{R}$. [50 marks]

   (c) Let V be a vector space over a field $\mathbb{F}$ and W be a non-empty subset of V. Prove that W is a subspace of V, if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in \mathbb{F}$. [20 marks]

   (d) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices over the field $\mathbb{R}$, and let $W = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$. Is W a subspace of $M_{2\times 2}$? Justify your answer.

   [20 marks]

2. (a) Define the following:

i. a linearly independent set of vectors;

ii. a basis for a vector space;

iii. direct sum of two subspaces S and W of a vector space V. [15 marks]

(b) i. Let S and W be two subspaces of a vector space V over the field $\mathbb{F}$. Prove that V is a direct sum of S and W if and only if each vector $v \in V$ has a unique representation $v = s + w$, for some $s \in S$ and $w \in W$.

ii. Let U and W be two subspaces of $\mathbb{R}^3$ defined by

$U = \{(a, b, c) | \ a = b = c, \text{ and } a, b, c \in \mathbb{R}\}$ and $W = \{(0, p, q) | \ p, q \in \mathbb{R}\}$.

Show that $\mathbb{R}^3 = U \oplus W$. [35 marks]

(c) Let S be any non-empty linearly independent subset of an n-dimensional vector space V over the field $\mathbb{F}$. Prove that

i. for any $v \in V$ the set $S \cup \{v\}$ is linearly independent if and only if $v \notin \langle S \rangle$.

ii. any linearly independent set of vectors of V can be extended to a basis of V.

Hence extend the subset $\{(1, 2, 1), (3, -4, 7)\}$ to a basis for $\mathbb{R}^3$.

[50 marks]

3. (a) Define the following terms:

i. Range space $R(T)$;

ii. Null space $N(T)$;

of a linear transformation T from a vector space V into another vector space W. [10 marks]

Prove that the image of any linearly independent subset of V is a linearly independent subset of W if and only if $N(T) = \{0\}$. [20 marks]

(b) Find $R(T)$ and $N(T)$ of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z)$, for any $(x, y, z) \in \mathbb{R}^3$.

Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$, for this linear transformation. [30 marks]

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$T(x, y, z) = (2x + y + 3z, 3x - y + z, -4x + 3y + z)$, for any $(x, y, z) \in \mathbb{R}^3$.

Let $B_1 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ and $B_2 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ be bases for $\mathbb{R}^3$. Find

i. the matrix representation of T with respect to the basis B_1;

ii. the matrix representation of T with respect to the basis B_2 by using the transition matrix. [40 marks]

4. (a) Define the following terms as applied to a matrix:

i. Rank;

ii. Echelon form;

iii. Row reduced echelon form. [15 marks]

(b) Let A be a $n \times n$ non-singular matrix. Prove that

i. A is row equivalent to I;

ii. A can be written as a product of elementary matrices;

iii. $r(BA) = r(B)$, for every $n \times n$ matrix B. [40 marks]

(c) Let r be the rank of the matrix A given by

$$A = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ -\beta & 1 & \beta & 0 \\ 0 & -\gamma & 1 & \gamma \\ 0 & -\delta & 1 & \delta \end{pmatrix}.$$

Prove that

i. $r = 2$ if and only if $\alpha\beta = -1$ and $\gamma = \delta = 0$;

ii. $r = 3$ if and only if either $\alpha\beta = -1$ or $\gamma = \delta$, here γ and δ are not both zero. [45 marks]

5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

i. Cofactor A_{ij} of an element a_{ij};

ii. Adjoint of A. [10 marks]

Prove that

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = (\det A) \cdot I,$$

where I is the $n \times n$ identity matrix. [40 marks]

(b) Let J be the $n \times n$ real matrix with every entry equals to one, so that $J^2 = nJ$, and let $A = \alpha I_n + \beta J$, where $\alpha, \beta \in \mathbb{R}$.

i. Show that $\det A = \alpha^{n-1}(\alpha + n\beta)$.

ii. If $\alpha \neq 0$ and $\alpha \neq -n\beta$, prove that A is non-singular by finding an inverse for it of the form $\frac{1}{\alpha}(I_n + \rho J)$, where I_n is the identity matrix of order n and ρ any real number.

Hence find the inverse of the matrix

$$\begin{pmatrix} 6 & 4 & 4 & 4 \\ 4 & 6 & 4 & 4 \\ 4 & 4 & 6 & 4 \\ 4 & 4 & 4 & 6 \end{pmatrix}.$$

[50 marks]

6. (a) State the necessary and sufficient condition for a linear system of equations to be consistent. [10 marks]

(b) Show that the linear system of equations,

$$\begin{aligned} x_1 - 3x_2 + x_3 + cx_4 &= b, \\ x_1 - 2x_2 + (c-1)x_3 - x_4 &= 2, \\ 2x_1 - 5x_2 + (2-c)x_3 + (c-1)x_4 &= 3b + 4, \end{aligned}$$

is consistent, for all values of b if $c \neq 1$, where $a, b \in \mathbb{R}$. Find the value of b for which the system is consistent if $c = 1$, and the general solution when b has this value and $c = 1$. [50 marks]

(c) State Crammer's rule for 3×3 matrix and use it to solve the following linear system of equations:

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 7; \\ x_1 + 2x_2 - 4x_3 &= 3; \\ 3x_1 - 4x_2 - 6x_3 &= 5. \end{aligned}$$

[40 marks]