



EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE -2010/2011

FIRST SEMESTER (APRIL, 2013)

PM 201 - VECTOR SPACES AND MATRICES

(PROPER & REPEAT)

Answer all Questions

Time: Three hours

1. (a) Define what is meant by

i. a vector space,

ii. a subspace of a vector space.

[10 marks]

(b) Let $M_{m \times n}$ be the set of all real $m \times n$ matrices. For any two matrices

$A = (a_{ij}), B = (b_{ij}) \in M_{m \times n}$, and for any $\lambda \in \mathbb{R}$, define an addition \oplus and scalar multiplication \odot as follows:

$$(a_{ij}) \oplus (b_{ij}) = (a_{ij} + b_{ij}),$$

$$\lambda \odot (a_{ij}) = (\lambda a_{ij}).$$

Prove that $(M_{m \times n}, \oplus, \odot)$ is a vector space over the field \mathbb{R} .

[50 marks]

(c) Let V be a vector space over a field \mathbb{F} and W be a non-empty subset of V .

Prove that W is a subspace of V , if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in \mathbb{F}$.

[20 marks]

(d) Let $M_{2 \times 2}$ be the vector space of all 2×2 matrices over the field \mathbb{R} , and let

$W = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$. Is W a subspace of $M_{2 \times 2}$? Justify your answer.

[20 marks]

2. (a) Define the following:

- i. a linearly independent set of vectors;
- ii. a basis for a vector space;
- iii. direct sum of two subspaces S and W of a vector space V . [15 marks]

(b) i. Let S and W be two subspaces of a vector space V over the field \mathbb{F} . Prove that V is a direct sum of S and W if and only if each vector $v \in V$ has a unique representation $v = s + w$, for some $s \in S$ and $w \in W$.

ii. Let U and W be two subspaces of \mathbb{R}^3 defined by

$$U = \{(a, b, c) \mid a = b = c, \text{ and } a, b, c \in \mathbb{R}\} \text{ and } W = \{(0, p, q) \mid p, q \in \mathbb{R}\}.$$

Show that $\mathbb{R}^3 = U \oplus W$. [35 marks]

(c) Let S be any non-empty linearly independent subset of an n -dimensional vector space V over the field \mathbb{F} . Prove that

- i. for any $v \in V$ the set $S \cup \{v\}$ is linearly independent if and only if $v \notin \langle S \rangle$.
- ii. any linearly independent set of vectors of V can be extended to a basis of V .

Hence extend the subset $\{(1, 2, 1), (3, -4, 7)\}$ to a basis for \mathbb{R}^3 .

[50 marks]

3. (a) Define the following terms:

i. Range space $R(T)$;

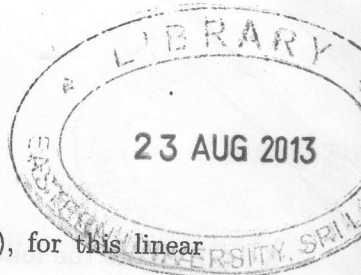
ii. Null space $N(T)$;

of a linear transformation T from a vector space V into another vector space W . [10 marks]

Prove that the image of any linearly independent subset of V is a linearly independent subset of W if and only if $N(T) = \{0\}$. [20 marks]

(b) Find $R(T)$ and $N(T)$ of a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, defined by

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z), \text{ for any } (x, y, z) \in \mathbb{R}^3.$$



Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$, for this linear transformation.

[30 marks]

(c) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T(x, y, z) = (2x + y + 3z, 3x - y + z, -4x + 3y + z), \text{ for any } (x, y, z) \in \mathbb{R}^3.$$

Let $B_1 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ and $B_2 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ be bases for \mathbb{R}^3 . Find

- i. the matrix representation of T with respect to the basis B_1 ;
- ii. the matrix representation of T with respect to the basis B_2 by using the transition matrix.

[40 marks]

4. (a) Define the following terms as applied to a matrix:

- i. Rank;
- ii. Echelon form;
- iii. Row reduced echelon form.

[15 marks]

(b) Let A be a $n \times n$ non-singular matrix. Prove that

- i. A is row equivalent to I ;
- ii. A can be written as a product of elementary matrices;
- iii. $r(BA) = r(B)$, for every $n \times n$ matrix B .

[40 marks]

(c) Let r be the rank of the matrix A given by

$$A = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ -\beta & 1 & \beta & 0 \\ 0 & -\gamma & 1 & \gamma \\ 0 & -\delta & 1 & \delta \end{pmatrix}.$$

Prove that

- i. $r = 2$ if and only if $\alpha\beta = -1$ and $\gamma = \delta = 0$;
- ii. $r = 3$ if and only if either $\alpha\beta = -1$ or $\gamma = \delta$, here γ and δ are not both zero.

[45 marks]

5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

i. Cofactor A_{ij} of an element a_{ij} ;

ii. Adjoint of A .

[10 marks]

Prove that

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = (\det A) \cdot I,$$

where I is the $n \times n$ identity matrix.

[40 marks]

(b) Let J be the $n \times n$ real matrix with every entry equals to one, so that $J^2 = nJ$, and let $A = \alpha I_n + \beta J$, where $\alpha, \beta \in \mathbb{R}$.

i. Show that $\det A = \alpha^{n-1}(\alpha + n\beta)$.

ii. If $\alpha \neq 0$ and $\alpha \neq -n\beta$, prove that A is non-singular by finding an inverse for it of the form $\frac{1}{\alpha}(I_n + \rho J)$, where I_n is the identity matrix of order n and ρ any real number.

Hence find the inverse of the matrix

$$\begin{pmatrix} 6 & 4 & 4 & 4 \\ 4 & 6 & 4 & 4 \\ 4 & 4 & 6 & 4 \\ 4 & 4 & 4 & 6 \end{pmatrix}.$$

[50 marks]

6. (a) State the necessary and sufficient condition for a linear system of equations to be consistent.

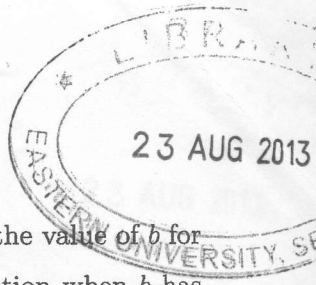
[10 marks]

(b) Show that the linear system of equations,

$$x_1 - 3x_2 + x_3 + cx_4 = b,$$

$$x_1 - 2x_2 + (c-1)x_3 - x_4 = 2,$$

$$2x_1 - 5x_2 + (2-c)x_3 + (c-1)x_4 = 3b+4,$$



is consistent, for all values of b if $c \neq 1$, where $a, b \in \mathbb{R}$. Find the value of b for which the system is consistent if $c = 1$, and the general solution when b has this value and $c = 1$. [50 marks]

(c) State Cramer's rule for 3×3 matrix and use it to solve the following linear system of equations:

$$2x_1 - 5x_2 + 2x_3 = 7;$$

$$x_1 + 2x_2 - 4x_3 = 3;$$

$$3x_1 - 4x_2 - 6x_3 = 5.$$

[40 marks]

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Prove that $(M_{m \times n}, \oplus, \odot)$ is a vector space over the field \mathbb{R} . [50 marks]

(c) Let V be a vector space over a field F and W be a non-empty subset of V . Prove that W is a subspace of V , if $u+v \in W$ for every $u, v \in W$ and for every $\alpha \in F$. [50 marks]

(d) Let $M_{2 \times 2}$ be the vector space of all 2×2 matrices over the field \mathbb{R} and let

$$W = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ be a subset of } M_{2 \times 2}. \text{ Justify your answer. [50 marks]}$$