EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE -2010/2011

SECOND SEMESTER (June, 2013)

PM 102 - REAL ANALYSIS

(PROPER & REPEAT)

Answer all Questions — Time: Three hours

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1. (a) i. Define the following terms of a non-empty subset of $\mathbb{R}$:

   - bounded above;
   - bounded below;
   - Supremum;
   - Infimum.

   ii. State the completeness property of $\mathbb{R}$.

   (b) Find the supremum, infimum, maximum and minimum of the following subsets of $\mathbb{R}$, or indicate where they do not exist:

   i. $[2,7) \setminus (2,4]$;

   ii. $\left\{\frac{\sqrt{n}}{1+n} : n \in \mathbb{N}\right\}$;

   iii. $\left\{\frac{1}{2^m} + \frac{1}{3^n} : m, n \in \mathbb{N}\right\}$.

   (c) Show that $\sqrt{3} - \sqrt{2}$ is not a rational number.

   (d) State the Mathematical induction principle and use it to prove that

   $$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \quad \forall\, n \in \mathbb{N}.$$

2. (a) Define the following terms:

   i. a sub sequence of a sequence;

   ii. Cauchy sequence.

   (b) State the Bolzano-Weierstrass theorem and use it to prove that a sequence of real numbers is Cauchy if and only if it is convergent.

   (c) i. Show that the sequence (x_n) given by $x_n = \sum_{i=1}^{n} \frac{1}{i}$ is not convergent.

   ii. Let (x_n) be a sequence of real numbers given by

$$x_1 = \frac{1}{1!}, \quad x_2 = \frac{1}{1!} - \frac{1}{2!}, \cdots, x_n = \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}, \cdots.$$

   Prove that (x_n) is convergent.

3. (a) Give the formal definition of the notion of a sequence of real numbers (x_n) converging to a limit l.

   Use this definition of limit of a sequence to establish the following limits:

   i. $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$;

   ii. $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$.

   (b) State the monotone convergence theorem.

   Let (x_n) be a sequence of real numbers defined inductively by

$$x_1 = 4, \quad x_{n+1} = \frac{1}{10}(x_n^2 + 21), \quad n \in \mathbb{N}.$$

   Show that

   i. $3 < x_n < 7$ for all $n \in \mathbb{N}$.

   ii. (x_n) is a decreasing sequence.

   Use the monotone convergence theorem to show that (x_n) is convergent and find its limit.

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. What is meant by the function f has a limit $l \in \mathbb{R}$ at a point 'a'($\in \mathbb{R}$).

Show that

$$\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that $\lim_{x \to a} f(x) = l$ if and only if for every sequence (x_n) that converges to 'a' such that $x_n \neq a$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to l.

Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions.

(a) State what is meant by the statement that f is differentiable at the point $a(\in \mathbb{R})$.

Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2 + 3x$ is differentiable at $x = 2$ and find the derivative of f at $x = 2$.

(b) Show that if f is differentiable at $x = a$ then f is continuous at $x = a$.

(c) Let f and g be differentiable at $x = a$. Show that $(f \cdot g)$ is differentiable at $x = a$ and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a)g'(a)$.

(d) If f is differentiable at 'a' and g is differentiable at '$f(a)$' then show that the composition function $g \circ f$ is differentiable at 'a' and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

6. (a) State the Intermediate value theorem for a continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a function with $a, b \in \mathbb{R}$ and $a < b$. Suppose that f is continuous on $[a, b]$ and that f is differentiable on (a, b). Show that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(You may use the Rolle's Theorem without proving it.)

(c) Let $f : [0, 1] \to \mathbb{R}$ be a function. Suppose that

- f is continuous on $[0, 1]$;
- f is differentiable on $(0, 1)$; and
- there exists $x \in (0, 1)$ such that $f(x) \neq x$.

Prove the following:

i. There exists $a \in (0, 1)$ such that $f(a) = \frac{1}{2}$.

ii. There exists $b \in (0, 1)$ such that $f'(b) = 1$.

iii. There exists $c \in (0, 1)$ such that $f'(c) > 1$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If f is continuous on $[a, b]$, differentiable on (a, b) and $f'(x) = 0$ for all $x \in [a, b]$, prove that f is a constant function on $[a, b]$.