EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE ~~2008/2009~~ 2010/2011

THIRD YEAR SECOND SEMESTER (Mar./May, 2011)

EXTMT 301 - GROUP THEORY

(REPEAT)

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nswer all questions **Time : Three hours**

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1. Define the following terms:

- *group*;
- *subgroup*;
- *center of a group.*

(a) Let H be a non empty subset of a group G. Prove that H is a subgroup of G if only if $ab^{-1} \in H$ for all $a, b \in H$.

(b) Let H be a subgroup of a group G. Prove that $H^{-1} = H$.

(c) Prove that the intersection of any two subgroups of a group is a subgroup.

(d) Prove that the centre of a group G is a subgroup of G.

2. (a) State and prove *Lagrange's theorem* for a finite group G.

(b) In a group G, H and K are different subgroups of order p, p is prime. Show t $H \cap K = \{e\}$, where e is the identity element of G.

(c) Prove that in a finite group G, the order of each element divides order of G. He prove that $x^{|G|} = e, \quad \forall\ x \in G$.

(d) Prove that every group of order less than 6 is abelian.

(e) If every non-identity element of a group G has order 2, show that G is abelian.

3. (a) Define what is meant by the $p-$*group*.

Prove the following:

i. every subgroup of a $p-$group is a $p-$group;

ii. the homomorphic image of a $p-$group is a $p-$group.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G;

ii. G' is a normal subgroup of G;

iii. G/G' is abelian;

iv. if H is a normal subgroup of G then G/H is abelian if and only if $G' \subseteq H$.

4. (a) State and prove the *first isomorphism theorem.*

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove

i. $K \trianglelefteq H$;

ii. $\dfrac{H}{K} \trianglelefteq \dfrac{G}{K}$;

iii. $\dfrac{G/K}{H/K} \cong \dfrac{G}{H}$.

5. State what is meant by a *normal subgroup* of a group G.

(a) Let $\phi : G \longrightarrow G_1$ be a homomorphism of a group G onto a group G_1. Prove the following:

i. $\ker\phi = \{g \in G \mid \phi(g) = e_1\}$ is a normal subgroup of G, where e_1 is an identity element of G_1;

ii. if H is a normal subgroup of G, then $\phi(H)$ is a normal subgroup of G_1.

(b) Let G be a group. Prove that for any non-empty subset H of G, $N(H) = \{x \in G \mid xH = Hx\}$ is a subgroup of G.

For any subgroup H of G, prove the following:

i. H is a normal subgroup of $N(H)$;

ii. $N(H)$ is the largest subgroup of G in which H is normal;

iii. H is a normal subgroup of G if and only if $N(H) = G$.

6. (a) Define the following terms as applied to a permutation group:

i. *cyclic of order* r;

ii. *transposition*;

iii. *signature*.

(b) Prove that the permutation group on n symbols S_n is a finite group of order $n!$.

Is S_n abelian for $n > 2$? Justify your answer.

(c) Prove that every permutation in S_n can be expressed as a product of disjoint cycles.

(d) Express the permutation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}$$

as a product of disjoint cycles. Hence or otherwise determine whether σ is even or odd.