EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE (2010/2011)

THIRD YEAR FIRST SEMESTER (Apr./ May, 2017)

EXTMT 302 - COMPLEX ANALYSIS

Special Repeat

Answer all questions **Time: Three hours**

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1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.

   (b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   at $z_0 = x_0 + iy_0$, then the derivative $f'(z_0)$ exists.

   (c) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being **harmonic**.

   Find the harmonic conjugate of $y^3 - 3x^2y$.

2. (a) i. Define what is meant by a path $\gamma : [\alpha, \beta] \to \mathbb{C}$.

ii. For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\, dz$.

(b) Let $a \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$. Show that

$$\int_{C(a;r)} (z-a)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1, \end{cases}$$

where $C(a;r)$ denotes a positively oriented circle with center a and radius r.

(State but **do not prove** any results you may assume).

(c) State the **Cauchy's Integral Formula**.

By using the **Cauchy's Integral Formula** compute the following integrals:

i. $\displaystyle\int_{C(0;2)} \frac{z}{(9-z^2)(z+i)}\, dz$;

ii. $\displaystyle\int_{C(0;1)} \frac{1}{(z-a)^k\,(z-b)}\, dz$, where $k \in \mathbb{Z}$ $|a| > 1$ and $b < 1$.

3. (a) State the **Mean Value Property for Analytic Functions**.

(b) i. Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**.

ii. Prove **Liouville's Theorem**: If f is entire and

$$\frac{\max\{|f(t)| : |t| = r\}}{r} \to 0, \quad \text{as } r \to \infty,$$

then f is constant.

(State any result you use without proof).

iii. Prove the **Maximum-Modulus Theorem**: Let f be analytic in an open nected set A. Let γ be a simple closed path that is contained, together wi inside, in A. Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z)| = M$, then f is constant throu A. Consequently, if f is not constant in A, then

$$|f(z)| < M, \quad \forall z_0 \text{ inside } \gamma.$$

(State any result you use without proof)

. (a) Let $\delta > 0$ and let $f : D^*(z_0;\delta) \to \mathbb{C}$, where $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$.

Define what is meant by

i. f having a singularity at z_0;

ii. the order of f at z_0;

iii. f having a pole or zero at z_0 of order m;

iv. f having a simple pole or simple zero at z_0.

(b) Prove that $ord(f, z_0) = m$ if and only if

$$f(z) = (z - z_0)^m g(z), \quad \forall z \in D^*(z_0;\delta),$$

for some $\delta > 0$, where g is analytic in $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$ and $g(z_0) \neq 0$.

(c) Prove that if f has a simple pole at z_0, then

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z_0),$$

where $Res(f; z_0)$ denotes the residue of $f(z)$ at $z = z_0$.

5. (a) Let f be a analytic in the upper-half plane $\{z : Im(z) \geqslant 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leqslant \frac{M}{|z|^\alpha}, \quad |z| \geqslant R \quad \text{with} \quad \text{Im}(z) \geqslant 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2}\, dx.$$

(You may assume without proof the Residue Theorem)

6. (a) State the **Principle of Argument Theorem**.

(b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open starse

Suppose that

i. f, g are analytic in A except for finitely many poles, none lying on γ.

ii. f and $f + g$ have finitely many zeros in A.

iii. $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f + g; \gamma) = ZP(f; \gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denotes the number of zeros - number of p inside γ of $f+g$ and f respectively, where each is counted as many times a order.

(c) State the **Fundamental Theorem of Algebra**.