EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

ACADEMIC YEAR - 2013/2014 (June, 2016)

MTS 08 - RELATIVITY

nswer all questions — Time : Three hours

1. In two spacetime dimensions two observers moving with constant relative velocity v set up coordinate system (ct, x) and (ct', x') respectively. Show that if they set their clocks to $t = t' = 0$ when pass each other, the transform between these coordinate systems is the Lorenz transform:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \text{where } \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

Let K and K' be two inertial frames such that the origin of K' moves with relative speed v in the $x-$direction.

(a) In K a rod at rest has length l_0. What is the length of the rod in K'?

(b) Let A and B be two simultaneous events in K and suppose A is at $(0,0)$ and B is at $(0,x)$ where $x \neq 0$. Show that A and B are not simultaneous in K'.

(c) Show that a particle moving with the speed of light along the $x-$direction in K also moves at the speed of light in K'.

2. (a) Two particles with rest mass m_1 and m_2 are moving along $x-$direction with velocities u_1 and u_2. These particles collides and form a new particle with rest mass m_3 and also moving along the $x-$direction with velocity u_3. Show that

$$u_3 = \frac{m_1\gamma_1 u_1 + m_2\gamma_2 u_2}{m_1\gamma_1 + m_2\gamma_2} \text{ and } m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

where $\gamma_i = \gamma(u_i) = \left(1 - \dfrac{u_i^2}{c^2}\right)^{-1/2}$.

(b) Suppose that a photon is traveling along $x-$axis and collides with a statio[nary]
electron of mass m. After the collision the photon and electron move in [the] x
plane and making angles of θ (anti-clockwise) and ϕ (clockwise) with x-
respectively. Show that:

i.
$$\bar{\nu} = \frac{\nu}{1 + \left(\frac{h\nu}{mc^2}\right)(1 - \cos\theta)},$$

where ν and $\bar{\nu}$ are the frequencies of the photon before and after [the] collision and h is the Plank's constant.

ii.
$$\sin^2\frac{\theta}{2} = \frac{mc}{2}\left(\frac{1}{q} - \frac{1}{p}\right),$$

where p and q are the momentum of photon before and after the coll[ision].

3. (a) Show that the Riemann tensor

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}$$

arises from the equation $V_{a;bc} - V_{a;cb} = R^d{}_{abc}V_d$.

(b) Using the Bianchi identity

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$$

show that $G^{ab}{}_{;b} = 0$.

(c) Prove the following:

i. $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$;

ii. $\lambda_{a;bc} = R_{abcd}\lambda^d$ if $\lambda_{a;b} + \lambda_{b;a} = 0$.

4. Use the Euler-Lagrange equations to obtain the non-vanishing Christoffel s[ymbols] for the metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Hence show that the only non-zero Ricci tensor components for this metric ar[e given] by

$$\begin{aligned}
R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} \\
R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} \\
R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \\
R_{33} &= R_{22}\sin^2\theta,
\end{aligned}$$

where $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and prime denotes the differentiation with respect to r.

5. (a) Using the result obtained in question 4, derive the vacuum solution of the Einstein field equations for the static, exterior geometry of a massive object

$$ds^2 = -c^2\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

clearly stating all results used

(b) Let the equations for a particle be

$$\begin{aligned} \left(1 - \frac{2m}{r}\right)\dot{t} &= k \\ r^2\dot{\phi} &= h \\ c^2\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 &= c^2. \end{aligned}$$

Show that the following results hold in vertical free fall:

$$\begin{aligned} k &= \sqrt{1 - 2m/r_0} \\ \ddot{r} + \frac{mc^2}{r^2} &= 0 \\ \frac{1}{2}\dot{r}^2 &= mc^2\left(\frac{1}{r} - \frac{1}{r_0}\right) \end{aligned}$$

where r_0 is the point of release of the particle.

6. (a) Let u be an affine parameter along the null geodesic of the light pulse. Show that the change in coordinate time can be written as

$$\frac{dt}{du} = \frac{1}{c}\left[\left(1 - \frac{2m}{r}\right)^{-1} g_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\right]^{1/2},$$

which integrates to give

$$t_R - t_E = \frac{1}{c}\int_{u_E}^{u_R}\left[\left(1 - \frac{2m}{r}\right)^{-1} g_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\right]^{1/2} du,$$

where E and R denotes the emitter and receiver respectively.

(b) Using the expression above explain why the change in coordinate time between the time of emission between consecutive signals (Δt_E) is the same as the change in coordinate time between time of reception (Δt_R) of these signals? Using this fact, argue that the change in proper time between emission and reception is given by

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1 - \frac{2m}{r_R}}{1 - \frac{2m}{r_E}}\right)^{1/2}.$$

(c) If the pulses are emitted and received with frequencies $\nu_e = n/\Delta\tau_E$ and ν $n/\Delta\tau_R$, show that in the limit of r/m being small the spectral shift is give

$$\frac{\Delta\nu}{\nu_E} \equiv \frac{\nu_R - \nu_E}{\nu_E} \approx \frac{GM}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right).$$

(d) The wavelength of a helium-neon lases is measured inside a *Skylab* freely f ing far out in deep space, and is found to be 632.8 nm. What wavelength w an experimenter measure if:

i. he and the laser fell freely together towards a neutron star?

ii. he remained in the freely floating *Skylab* while the laser transmitted ra from the surface of the neutron star of mass 10^{30}kg and radius $r_B =$ 1

iii. he were beside the laser, both on the surface of the neutron star?

iv. he were on the surface of neutron star while the laser was back i distant *Skylab*?