EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
THIRD EXAMINATION IN SCIENCE - 2014/2015
SECOND SEMESTER (Dec., 2017/Jan., 2018)
PM 301 - GROUP THEORY

nswer all questions — Time : Three hours

1. (a) Define a *group*. (*10 Marks*)

   (b) Let G be a group and let $a, b, c \in G$. Prove the following:

      i. if $ab = ac$, then $b = c$,

      ii. if $ac = bc$, then $a = b$. (*20 Marks*)

   (c) Let S be the set of all real numbers except -1. Define an operation $*$ on S by

   $$a * b = a + b + ab$$

   for each $a, b \in S$. Show that $(S, *)$ is a group.
   Is S an abelian group with this operation? Justify your answer. (*40 Marks*)

   (d) Let G be a group and suppose that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Show that G is an abelain group. (*30 Marks*)

2. (a) Define a *cyclic group*. (*10 Marks*)

   (b) Prove that if a is a generator of a cyclic group, then a^{-1} is also a generator of that group. (*20 Marks*)

   (c) Let $G = \{1, 2, 3, 4\}$ be a group with the binary operation "multiplication modulo 5". By using the Cayley's table or otherwise, show that G is a group with this operation. Also show that G is cyclic and find all the generators of G. (*40 Marks*)

   (d) Find all the cyclic subgroups generated by the elements $(1, 2, 3)$, $(1, 3, 2)$ and $(2, 3)$ in the group S_3. (*30 Marks*)

3. (a) Let H be a subgroup of a group G. Prove the following:

   i. The identity element of H is the same of the identity element of G. *(20 Marks)*

   ii. The inverse of an element a in H is the same as the inverse when we consider a in G. *(20 Marks)*

   (b) Let $H = \{2^k \mid k \in \mathbb{Z}\}$. Show that H is a subgroup of $\mathbb{Q}^*$, where $\mathbb{Q}^*$ is the group of non-zero rational numbers with the usual multiplication.

   Find the identity element of H and the inverse of an element a in H and verify the results in (a) above. *(40 Marks)*

   (c) Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5.

   (20 Marks)

4. (a) Let $\phi : G \longrightarrow G'$ be a homomorphism between the groups G and G'. Prove that if K is a subgroup of G, then $\phi(K)$ is a subgroup of G'. *(30 Marks)*

   (b) Let $\mathbb{R}^*$ be the group of non-zero real numbers with usual multiplication as the binary operation. Define a mapping $\phi : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ by $\phi(x) = x^2$ for each $x \in \mathbb{R}^*$. Show that ϕ is a homomorphism and find the kernel of ϕ. *(30 Marks)*

   (c) Let G be a group and let $g \in G$. Let $\phi_g : G \rightarrow G$ be defined by $\phi_g(x) = gx$ for $x \in G$. For which $g \in G$ is ϕ_g a homomorphism? *(20 Marks)*

   (d) An *automorphism* of a group G is an isomorphism with itself. Show that the map $i_g : G \rightarrow G$ defined by $i_g(x) = gxg^{-1}$ is an automorphism of G, where G is a group and $g, x \in G$. *(20 Marks)*

5. (a) Define the *left cosets* of a subgroup in a group. *(10 Marks)*

   (b) Let H be a subgroup of a group G. Prove that the left cosets of H in G partition G, that is, the group G is the disjoint union of the left cosets of H in G. *(30 Marks)*

   (c) Find the partition of $\mathbb{Z}_6$ into the left cosets of the subgroup $H = \{0, 3\}$. *(30 Marks)*

   (d) Find all cosets of the subgroup $\langle 4 \rangle$ in the group Z_{12}. Also find the index of $\langle 4 \rangle$ in Z_{12}. *(30 Marks)*

(a) Let N be a normal subgroup of a group G. Prove that the set $G/N = \{aN \mid a \in G\}$ of left cosets of N in G is a group under the operation $(aN)(bN) = abN$ defined on it (this group is called the factor group of G and N). *(30 Marks)*

(b) If G is abelian, prove that G/N is also abelian. *(20 Marks)*

(c) Let $G = \mathbb{Z}_{18}$ be a group with addition modulo 18 and $N = \{0, 6, 12\}$ be a normal subgroup of G. Find the factor group of G and N. *(30 Marks)*

(d) Let G be the set of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and let H be the set the matrices of the form $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, where $a, b \in \mathbb{R}$. If G is a group under the matrix multiplication and H is a subgroup of G, then show that H is a normal subgroup of G. *(20 Marks)*