EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
THIRD EXAMINATION IN SCIENCE - 2014/2015
SECOND SEMESTER (October, 2017)
PM 301 - GROUP THEORY

Answer all questions | Time : Three hours

1. (a) Define the term *group*. (*10 Marks*)

   (b) In a group, prove the following:

   i. the identity element is unique, (*20 Marks*)

   ii. the inverse of an element is unique, (*20 Marks*)

   iii. the cancellation laws hold. (*20 Marks*)

   (c) Let $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Show that G is a group under addition.

   (*30 Marks*)

2. (a) Let G be a group and let H be a subset of G. Prove that H is a subgroup of G if and only if $ab^{-1} \in H$ for each $a, b \in H$. (*30 Marks*)

   (b) Show that the identity element of a group G and the identity element of the subgroups of G are the same. (*20 Marks*)

   (c) Show that the inverse of an element of a subgroup and the inverse of the element regarded as a member of the group are the same. (*20 Marks*)

   (d) Let G be a group and let $a \in G$. Then show that $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G. (*30 Marks*)

3. (a) i. Define the term *cyclic group* of a group. (*10 Marks*)

   ii. Show that every cyclic is abelian. (*20 Marks*)

   iii. Find the generators of the groups $\mathbb{Z}$ and $\mathbb{Z}_6$. (*20 Marks*)

(b) i. Let $A = \{1, 2, \cdots, n\}$ and let S_n be the set of all permutations on A. Show that S_n is group under permutation multiplication. *(30 Marks)*

ii. Express the following permutations as product of cycles: *(20 Marks)*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}.$$

4. (a) Let $\phi : G \to G'$ be an homomorphism between the groups G and G'. Then, prove the following:

   i. if e is the identity of G, then $\phi(e)$ is the identity of G', *(20 Marks)*

   ii. for any $g \in G$, $\phi(g^{-1}) = \phi(g)^{-1}$, *(20 Marks)*

   iii. if K is a subgroup of G, then $\phi(K)$ is a subgroup of G'. *(30 Marks)*

   (b) Show that the following mappings are homomorphisms: *(30 Marks)*

   i. $\phi : \mathbb{Z} \to G$ defined by $\phi(n) = g^n$, where G is a group and $g \in G$.

   ii. $\rho : G \to H$ defined by $\phi(x) = e^x$, where $G = \mathbb{R}$ is a group under addition and $H = \mathbb{R}^+$ is a group under multiplication.

5. (a) Let H be a subgroup of a group G and $a, b \in G$. Then prove the following:

   i. if $a \in H$ then $Ha = H$, *(20 Marks)*

   ii. if $ab^{-1} \in H$ then $Ha = Hb$. *(20 Marks)*

   (b) Let H be a subgroup of G and let $g \in G$. Show that the map $\Phi : H \to gH$ defined by $\Phi(h) = gh$ is bijective. *(30 Marks)*

   (c) Find the left cosets of the following subgroups H in the groups G: *(30 Marks)*

   i. if $H = 3\mathbb{Z}$ and $G = \mathbb{Z}$,

   ii. if $H = \{0, 3\}$ and $G = Z_6$,

   iii. if $H = \{(1), (123), (132)\}$ and $G = S_3$

6. (a) i. Define the *factor group* of a group. *(10 Marks)*

   ii. Prove that a factor group of a cyclic group is cyclic. *(30 Marks)*

   iii. Find the factor group $\mathbb{Z}/3\mathbb{Z}$. *(10 Marks)*

   (b) i. Define the *Lagrange's theorem.* *(10 Marks)*

   ii. If G is a finite group and $g \in G$, then prove that the order of g must divide the order of G. *(20 Marks)*

   iii. Let the order of the group G be p, where p is a prime. If $g \neq e$ is an element of G, then prove that G is a cyclic group generated by g. *(20 Marks)*