EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2015/2016

FIRST SEMESTER (May/June, 2018)

PM 302 - COMPLEX ANALYSIS

Answer all questions — Time: Three hours

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1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.

(b) Let the function $f(z) = u(x, y)+iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $z_0 = x_0 + iy_0$, then the derivative $f'(z_0)$ exists.

(c) i. Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being **harmonic**.

ii. Obtain a harmonic conjugate $v(x, y)$ of a harmonic function $u(x, y) = \dfrac{y}{x^2 + y^2}$ such that $f(x) = u(x, y) + iv(x, y)$ is analytic.

2. (a) i. Define what is meant by a **path** $\gamma : [\alpha, \beta] \to \mathbb{C}$.

ii. For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\, dz$.

(b) Prove that if $w(t)$ is a continuous complex valued function of t such that $\alpha < t < \beta$, then

$$\left| \int_\alpha^\beta w(t)\, dt \right| \leq \int_\alpha^\beta |w(t)|\, dt.$$

(c) Prove that if γ is a path and f be a continuous function on γ, then $|f(z)| \leq M$, for all $z \in \gamma$ and $M \geq 0$ such that $\left| \int_\gamma f(z) dz \right| \leq ML$, where $L = \text{Length}(\gamma)$.

Hence show that

$$\left| \int_\gamma \frac{z^{1/2}}{z^2+1} \right| \leq \frac{3\sqrt{3}\pi}{8},$$

where γ is the semi circular path given by $z = 3e^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

3. (a) Let $D(a; r) := \{z \in \mathbb{C} : |z - a| < r\}$ denotes the open disc with center $a \in \mathbb{C}$ and radius $r > 0$ and let f be analytic on $D(a; r)$ and $0 < s < r$. Prove **Cauchy's Integral Formula**,

$$f(z_0) = \frac{1}{2\pi i} \int_{C(a;s)} \frac{f(z)}{z - z_0}\, dz, \qquad \text{for } z_0 \in D(a; s),$$

where $C(a; s)$ denotes the circle with center a and radius $s > 0$.

(b) Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz \quad (|w| \neq 3),$$

then $g(2) = 8\pi i$. Find the value of $g(w)$ when $|w| > 3$?

4. (a) Let $\delta > 0$ and let $f : D^*(z_0;\delta) \to \mathbb{C}$, where $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$.

Define what is meant by f has a pole of order m at z_0.

(b) Prove that if $\operatorname{ord}(f, z_0) = m$ then $f(z) = (z - z_0)^m \, g(z)$, $\forall z \in D^*(z_0;\delta)$, for some $\delta > 0$, where g is analytic in $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$ and $g(z_0) \neq 0$.

(c) Prove that if f has a pole of order m at z_0, then

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} h(z) \right\}, \quad \text{where } h(z) = (z - z_0)^m \, f(z).$$

Show that the residue of the function $f(z) = \dfrac{\log z}{(z^2+1)^2}$ at i is $\dfrac{\pi + 2i}{8}$.

5. Let f be a analytic in the upper-half plane $\{z : \operatorname{Im}(z) \geqslant 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leqslant \frac{M}{|z|^\alpha}, \quad |z| \geqslant R \quad \text{with } \operatorname{Im}(z) \geqslant 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx.$$

(You may assume without proof the Residue Theorem).

6. (a) State the **Principle of Argument Theorem.**

(b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open starset A. Suppose that

i. f, g are analytic in A except for finitely many poles, none lying on γ.

ii. f and $f+g$ have finitely many zeros in A.

iii. $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denotes the excess number of zeros over poles of $f+g$ and f inside γ respectively, where each is counted as many times as its order.

(c) State the **Fundamental Theorem of Algebra.**

(d) Determine the number of zeros of $z^4 - 2z^3 + 9z^2 + z - 1$ in the circle $|z| = 2$.