EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2014/2015

SECOND SEMESTER (Dec.,2017/Jan., 2018)

PM 303 - FUNCTIONAL ANALYSIS-I

Answer all questions **Time : Two hours**

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1. Define the term *norm* on a linear space. (*10 Marks*)

   (a) Show that the following formula

   $$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

   defines a norm for x in the l^p space, where $1 < p < \infty$. (*30 Marks*)

   (b) A norm $\|\cdot\|$ on a linear space X is said to be equivalent to a norm $\|\cdot\|_0$ on X if

   $$a\,\|x\|_0 \leq \|x\| \leq b\,\|x\|_0$$

   for each $x \in X$, where a and b are positive numbers.

   On a finite dimensional vector space, show that any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$ (*30 Marks*)

   (c) Let

   $$\|x\|_1 = \sum_{1=1}^{n} |x_i|, \quad \text{and} \quad \|x\|_\infty = \max_{1\leq i\leq n} |x_i|$$

   be two norms on a linear space $X = \mathbb{F}^n$, where $x = (x_1, x_2, \cdots, x_n)$. Show that $\|x\|_1$ is equivalent to $\|x\|_\infty$ in the space X. (*30 Marks*)

2. Define a *linear operator* between two normed linear spaces. (*10 Marks*)

(a) Let T be a linear operator from a vector spaces X to a vector space Y. Prove the following:

i. The range of T, $R(T)$, is a vector space. (*20 Marks*)

ii. The inverse operator of T, $T^{-1} : R(T) \longrightarrow D(T)$ exists if and only if

$$Tx = 0 \Longrightarrow x = 0,$$

where $D(T)$ is the domain of T. (*30 Marks*)

iii. If T^{-1} exists, then it is a linear operator. (*20 Marks*)

(b) Show that

$$Tx(t) = \int_a^t x(\tilde{c})d\tilde{c},$$

$t \in [a, b]$, is a linear operator on the space of continuous functions $C[a, b]$. (*20 Marks*)

3. Let X and Y be normed linear spaces. The norm of a bounded linear operator T from the domain of T, $D(T)$, to Y is given by

$$\|T\| = \sup_{x \in D(T), x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

(a) If X is a finite dimensional normed linear space, then show that every linear operator on X is bounded. (*20 Marks*)

(b) Prove that if a linear operator is continuous then it is bounded. (*30 Marks*)

(c) Let T be a bounded linear operator from a normed linear space X to a normed linear space Y.

i. Show that the null space of T is closed. (*20 Marks*)

ii. Prove that T is bounded if and only if T maps bounded sets in X into a bounded sets in Y. (*30 Marks*)

4. (a) State the *Hahn-Banach theorem for normed linear spaces* and prove this by using the generalized Hahn-Banach theorem. (*40 Marks*)

(b) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional $\bar{f}$ on X such that

$$\|\bar{f}\| = 1 \quad \text{and} \quad \bar{f}(x_0) = \|x_0\|.$$

(*30 Marks*)

(c) For every x in a normed linear space X, show that

$$\|x\| = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|}$$

where X' is the dual space of X. (*30 Marks*)