# EASTERN UNIVERSITY, SRI LANKA

## XTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

## SECOND YEAR SECOND SEMESTER (April/May, 2016)

## EXTMT 204 - RIEMANN INTEGRAL AND SEQUENCES AND SERIES OF FUNCTIONS (REPEAT)

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nswer all Questions | Time: Two hours

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Q1. (a) Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function on $I$.

Define

(i) the *lower sum* of $f$;

(ii) the *upper sum* of $f$;

corresponding to a partition $P$ of $I$.

(b) (i) Prove that if $f \in \mathbb{B}[a, b]$, then for any partition $P$ of $[a, b]$ we have

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a);$$

(ii) Prove that for any $f \in \mathbb{B}[a, b]$, we have

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

(c) Let $f$ be a continuous function over $[a, b]$. Prove that an integral

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right).$$

Hence find the value of $\displaystyle\int_0^2 x^2\, dx$.

Q2. (a) Prove that, if $f$ is positive, continuous and monotonic decreas defined on $[1, \infty)$ such that $f(k) = a_k$ for each natural number sequence $\{a_k\}$ of positive terms is monotonic decreasing such that converges or diverges according to $\int_1^\infty f(x)\,dx$ converges or diverg

Using the above result, show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(b) State what is meant by the integral, $\int_a^b f(x)\,dx$, being an imprope the first kind.

Discuss the convergence of the following integral

$$\int_a^\infty \frac{1}{x^p}\,dx$$

where $p$ is a constant and $a > 0$.

Q3. (a) Prove that a sequence of real valued functions $\{f_n\}$ defined on $E \subseteq$ uniformly on $E$ if and only if for all $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such th $|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in E$ and $m, n \geq N_\varepsilon$.

(b) By using the result, that is, a sequence $\{f_n\}$ of functions on a set converges uniformly on $A$ to a limit function $f : A \to \mathbb{R}$ if and onl

$$\lim_{n\to\infty} [\sup |f_n(x) - f(x)| : x \in A] = 0,$$

show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$, $n \in \mathbb{N}$, co not uniformly on [0,1].

(c) Prove that if $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous function on a set suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $A$ to a function $f :$ $f$ is continuous on $A$.

Q4. Let $f_n$ be a sequence of real-valued functions defined on a subset $E$ $f : E \to \mathbb{R}$. Define what is meant by saying that the infinite series $\sum_{n=1}^{\infty}$ to $f$ uniformly on $E$.

Let $\{f_n\}$ and $\{g_n\}$ be two sequences of functions defined on a none Suppose also that,

1. $|S_n(x)| = \left|\sum_{k=1}^{n} f_k(x)\right| \leq M$ for all $x \in E$ and $n \in \mathbb{N}$;
2. $\sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$ converges uniformly in $E$;
3. $g_n \to 0$ uniformly in $E$.

Prove that $\sum_{k=1}^{\infty} f_k(x)g_k(x)$ converges uniformly in $E$.

Hence or otherwise show that, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+ax^2}$, $a > 0$ converges uniformly in $\mathbb{R}$.