EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST YEAR EXAMINATION IN SCIENCE - 2013/2014

SECOND SEMESTER - (APRIL/MAY, 2016)

PM 107 - THEORY OF SERIES

(PROPER & REPEAT)

Answer **All** Questions Time Allowed: **2** Hours

Q1. (a) Define what is meant by saying the infinite series of real numbers $\sum_{n=1}^{\infty} a_n$ is *convergent.* **[10 Marks]**

Let $a_n = 1/[(4n-1)(4n+3)]$ for all $n \in \mathbb{N}$, and let $s_n = a_1 + a_2 + \ldots + a_n$. Express a_n in partial fractions and hence, or otherwise, show that

$$s_n = \frac{1}{12} - \frac{1}{(4n+3)}.$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n+3)}$$

converges and find its sum. **[40 Marks]**

(b) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non negative real numbers such that $a_n \leq K\, b_n$ for all $n \in \mathbb{N}$ and for some number $K > 0$. Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. Using the result, or otherwise, determine the convergence of the series given by

$$\sum_{n=1}^{\infty} \frac{\ln n}{2n^3 - 1}.$$

[50 Marks]

Q2. (a) Define what it means to say that a series of real numbers is

(i) *absolutely convergent* ;

(ii) *conditionally convergent.*

[20 Marks]

(b) Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges so does $\sum_{n=1}^{\infty} a_n$. Making use of the result, or otherwise, determine the convergence of the following series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2\left(\frac{1}{n}\right).$$

[40 Marks]

(c) State the theorem of *Alternating Series Test.*

[10 Marks]

Use the theorem, to decide whether the following series converge or diverge

$$\sum_{n=1}^{\infty}\sin\left(\frac{(n^2+1)\pi}{n}\right).$$

[30 Marks]

Q3. Let a real-valued function f be continuous, decreasing and positive on $[1,\infty)$. Prove that the sequences (s_n) and (I_n) given by

$$s_n := f(1)+f(2)+\cdots+f(n)=\sum_{k=1}^{n} f(k),$$

$$I_n := \int_1^n f(x)\,dx, \quad n=1,2,3,\ldots,$$

are either both convergent or both divergent. [70 Marks]

Using the result, or otherwise, prove that

$$\frac{\pi}{4}<\sum_{n=1}^{\infty}\frac{1}{n^2+1}<\frac{(\pi+2)}{4}.$$

[30 Marks]

Q4. (a) For the power series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n}$, find the interval and radius of convergence.

[30 Marks]

(b) Show that

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} \quad \text{for } |x-1| < 1.$$

Using the result and the **Abel**'s theorem, show that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$

[40 Marks]

(c) (i) Let $z_n = x_n + iy_n$ for all $n \in \mathbb{N}$ and let $z = x + iy$. Prove that $\sum_{n=1}^{\infty} z_n = z$ if and only if $\sum_{n=1}^{\infty} x_n = x$ and $\sum_{n=1}^{\infty} y_n = y$.

[20 Marks]

(ii) Prove that if the series of complex numbers $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n \to \infty} |z_n| = 0$.

[10 Marks]

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