EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2012/2013

SECOND SEMESTER (Aug./Sept., 2015)

PM 102 - REAL ANALYSIS

(Proper & Repeat)

Answer all Questions | Time: Three hours

Q1. (a) i. Define the terms *supremum* and *infimum* of a non-empty subset of $\mathbb{R}$.

ii. State the *completeness* property of $\mathbb{R}$, and use it to prove that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

(b) Let S be non-empty subset of $\mathbb{R}$ which is bounded above and $a \in \mathbb{R}$. Let the set $a + S$ be defined as

$$a + S = \{a + x : x \in S\}.$$

Prove that $\sup(a + S) = a + \sup S$.

(c) Find the supremum and infimum of the set

$$S = \left\{\frac{2}{3}\left(1 - \frac{1}{10^n}\right) : \ n \in \mathbb{N}\right\}.$$

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Q2. (a) State what it means for a sequence of real numbers (x_n) to *converge* to a limit a.

Use the definition of convergence of sequence of real numbers to show that

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0.$$

(b) Prove that every increasing sequence, which is bounded above, is convergent. Deduce that every decreasing sequence, which is bounded below, is convergent.

(c) Let the sequence (x_n) be defined inductively by

$$x_1 = \frac{3}{2} \text{ and } x_{n+1} = 3 - \frac{2}{x_n}, \text{ for all } n \in \mathbb{N}.$$

i. Show that (x_n) is monotonic and bounded.

ii. Find its limit.

Q3. (a) Define the following terms:

i. a *subsequence* of a sequence;

ii. *Cauchy* sequence.

(b) If a real sequence (x_n) is such that $x_n \longrightarrow l(\in \mathbb{R})$, then every subsequence of (x_n) has the same limit l.

Use this result to prove that

$$\lim_{n\to\infty} a^n = 0 \text{ if } 0 < a < 1.$$

(c) i. Prove that $\left(\frac{1}{n^2}\right)$ is a Cauchy sequence.

ii. Prove that every Cauchy sequence is bounded.

Q4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what is meant by the function f has a limit $l(\in \mathbb{R})$ at a point $a(\in \mathbb{R})$.

(b) If $\lim_{x\longrightarrow a} f(x) = l$, then show that $\lim_{x\longrightarrow a} |f(x)| = |l|$. Is the converse true? Justify your answer.

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(c) i. Let $A \subseteq \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ be a function. Prove that $\lim_{x \longrightarrow a} f(x) = l$ if and only if for every sequence (x_n) in A with $x_n \longrightarrow a$ as $n \longrightarrow \infty$ such that $x_n \neq a$ for all $n \in \mathbb{N}$, we have $f(x_n) \longrightarrow l$ as $n \longrightarrow \infty$.

ii. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 1/x$ for all $x \neq 0$. Show that $\lim_{x \longrightarrow 0} f(x)$ does not exist in $\mathbb{R}$.

Q5. (a) Define what it means to say that a real-valued function f is continuous at a point a in its domain.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that, f is continuous at $x = 0$.

(b) Show that, if f is continuous at a and $f(a) > 0$ then there exists some $\delta > 0$ such that $f(x) > f(a)/2$ for all x satisfying $|x - a| < \delta$.

(c) Let $f : A \to \mathbb{R}$ be a function and let $c \in A (\subseteq \mathbb{R})$. Show that the following conditions are equivalent:

i. f is continuous at c;

ii. If (x_n) is a sequence in A such that (x_n) converges to c, then $\lim_{n \to \infty} f(x_n) = f(c)$.

Q6. (a) i. Define what it means to say that the real-valued function f is differentiable at a point a in its domain.

ii. Prove that every differentiable function is continuous. Is the converse true? Justify your answer.

(b) State the *Mean Value Theorem*, and use it to prove

$$x < \sin^{-1} x < \frac{x}{\sqrt{1 - x^2}}, \quad \text{for all } x \in (0, 1).$$

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(c) Suppose that f and g are continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

If $f(d) = g(d) = 0$ for some $d \in (a, b)$ deduce that

$$\lim_{x \to d} \frac{f(x)}{g(x)} = \lim_{x \to d} \frac{f'(x)}{g'(x)}.$$