82

EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2012/2013

SECOND SEMESTER (Aug./Sept., 2015)

PM 107 - THEORY OF SERIES

(PROPER & REPEAT)

Answer all questions Time: Two hours

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1. (a) Define what is meant by saying that a series of real numbers $\sum_{n=1}^{\infty} a_n$ is convergent.

   Determine the convergence of the following series, and find the sum of each of the series, if it exists:

   i. $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$;

   ii. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2\,(n+1)^2}$.

   (State, without proof, any test(s) that you may use.)

   [40 marks]

   (b) Show that the following series converges

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}.$$

   [20 marks]

   (c) State Abel's partial summation formula.

   Suppose that the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is bounded, and (b_n) is a sequence of bounded variation converges to 0. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

   Hence prove that the following series

   $$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2},$$

where $x \neq 2k\pi,\ k = 0, 1, 2, \cdots$, is convergent.

[40 marks]

2. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers such that

   i. $a_n \leq m\, b_n$ for all $n \in \mathbb{N}$ and for some positive real number m, and

   ii. $\sum_{n=1}^{\infty} b_n$ converges.

   Show that the series $\sum_{n=1}^{\infty} a_n$ converges.

   By using the above result, determine whether the following series converges or diverges

$$\sum_{n=2}^{\infty} \frac{3}{\ln n^2}.$$

   (You may assume, without proof, the analog of the above result for divergence.)

[35 marks]

   (b) State the Integral Test for the series of positive real numbers.

   Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \tan\left(\frac{1}{n}\right)$ converges.

   $\left(\text{You may use, without proof, the result, } \frac{d}{dx}\left[\ln\left(\cos\left(\frac{1}{x}\right)\right)\right] = \frac{1}{x^2}\tan\left(\frac{1}{x}\right)\right.$

[25 marks]

   (c) Investigate whether the following series is convergent or divergent by using the Alternating Series Test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}.$$

[20 marks]

   (d) Determine the convergence of the following series by using the limit form of the Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt{n+2}}.$$

[20 marks]

3. (a) For each of the following series, determine whether it is absolutely convergent or conditionally convergent.

   i. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2 + n^3}{n^4 + \ln n}\right)$;

84

ii. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$.

[40 marks]

(b) Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{1+n}}{n^2}(x-2)^n.$$

[30 marks]

(c) Find the power series representation of the function

$$f(x)=\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right).$$

(You may assume, without proof, the result, $\displaystyle\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ where a, c are arbitrary constants.)

[30 marks]

4. (a) Let $\displaystyle\sum_{n=1}^{\infty}b_n$ be a rearrangement of an absolutely convergent series $\displaystyle\sum_{n=1}^{\infty}a_n$. Prove that

i. $\displaystyle\sum_{n=1}^{\infty}b_n$ is absolutely convergent;

ii. $\displaystyle\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}a_n$.

(State, without proof, any result you may use.)

[25 marks]

(b) Given that, $\ln(1+x)=\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for $|x|<1$. Find the sum of the conditional convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

Rearrange the above series to obtain a different sum.

[35 marks]

(c) Let $\displaystyle\sum_{n=1}^{\infty}a_n$ be a conditionally convergent series. Prove that for every real number S, there is a rearrangement $\displaystyle\sum_{n=1}^{\infty}b_n$ of $\displaystyle\sum_{n=1}^{\infty}a_n$ such that $\displaystyle\sum_{n=1}^{\infty}b_n=S$.

[40 marks]