EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2012/2013

FIRST SEMESTER (~~March/April~~ Feb. Mar., 2016)

PM 201 - VECTOR SPACES AND MATRICES

Proper & Repeat

r all questions

Time: Three hours

(a) i. Define what is meant by the term *subspace* of a vector space.

Let V be a vector space over a field $\mathbb{F}$. Prove that a non-empty subset S of V is a subspace of V if and only if $ax + by \in S$ for any $x, y \in S$ and $a, b \in \mathbb{F}$.

ii. Which of the following sets are subspaces of $\mathbb{R}^2$ under usual addition and scalar multiplication? Justify your answer.

A. $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{Z} \right\}$;

B. $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 x_2 = 0, x_1, x_2 \in \mathbb{R} \right\}$.

(b) Let $V = \{f : f : \mathbb{R} \to \mathbb{R}, f(x) > 0, \forall x \in \mathbb{R}\}$. For any $f, g \in V$ and for any $\alpha \in \mathbb{R}$ define an addition $\oplus$ and a scalar multiplication $\odot$ as follows:

$$(f \oplus g)(x) = f(x).g(x)$$

and

$$(\alpha \odot f)(x) = (f(x))^{\alpha}$$

for any $x \in \mathbb{R}$. Prove that $(V, \oplus, \odot)$ is a vector space over the set of real numbers $\mathbb{R}$.

2. (a) Define what is meant by

    i. a *linearly independent* set of vectors;

    ii. a *basis* of a vector space;

   (b) Let V be a vector space. Show that

    i. Any linearly independent set of vectors of V may be extended to a ba[sis of]
    V;

    ii. If L is a subspace of V, then there exists a subspace M of V su[ch that]
    $V = L \oplus M$, where $\oplus$ denotes the direct sum.

    iii. Let S and T be two subspaces of a vector space of V over the field [...]
    that $S \cap T = \{0\}$. Prove that, if $\{s_1, s_2, \cdots, s_k\}$ and $\{t_1, t_2, \cdots, t_l\}$ are li[nearly]
    independent subsets of S and T, respectively, then $\{s_1, s_2, \cdots, s_k, t_1, t_2, \cdots\}$
    is a linearly independent subset of V.

   (c) State and prove *Stibnitz* replacement theorem for a vector space.
   Prove that for an n dimensional vector space V, if $\langle B \rangle = V$, then B is a ba[sis of]
   V, where $B = \{v_1, v_2, \cdots, v_n\} \subseteq V$.

3. (a) Define what is meant by the *range space* $R(T)$ and the *null space* $N(T)$ of a [linear]
   transformation T from a vector space V into another vector space W.
   Find $R(T)$ and $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z), \forall\, (x, y, z) \in \mathbb{R}.$$

   Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for the linear transfor[mation]
   T.

   (b) Let $B_1 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ [be]
   bases of $\mathbb{R}^3$.

    i. Find the matrix representation of T, which is defined in part (a), with r[espect]
    to the basis B_1;

    ii. Using the transition matrix, find the matrix representation of T with r[espect]
    to the basis B_2.

(a) Define what is meant by

(i) *rank* of a matrix;

(ii) *row reduced echelon form* of a matrix.

(b) Let A be an $m \times n$ matrix. Prove the following:

(i) row rank of A is equal to column rank of A;

(ii) if B is a matrix obtained by performing an elementary row operation on A, then A and B have the same rank.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 3 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 5 & 6 & 8 & -1 \\ 4 & 3 & 0 & 0 \\ 10 & 12 & 16 & -2 \\ 7 & 2 & 0 & 0 \end{pmatrix}.$$

Define what is meant by the term *adjoint of* A as applied to an $n \times n$ matrix $A = (a_{ij})$.

(a) With the usual notations, prove that

$$A \cdot (adj A) = (adj A) \cdot A = \det A \cdot I.$$

Hence prove that $\det(adj\ A) = (\det A)^{n-1}$

(b) Prove that, if $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, then $A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I$, where I is the 2×2 identity matrix and $n \in \mathbb{N}$.

(c) By applying appropriate row (column) operations, prove that the determinant of the matrix

$$\begin{pmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{pmatrix}$$

is $1 + a^4 + a^8$, where $a \in \mathbb{R}$.

6. (a) Let $n(\geq 2)$ be a positive integer and J be the $n \times n$ matrix in which entries is equal to 1. Show that $(I - J)^{-1} = I - \frac{1}{n-1}J$.

   (b) State the necessary and sufficient condition for a system of linear equa consistent.

   Investigate for what values of a, b following system of linear equations

   i. a unique solution

   ii. an infinite number of solutions

   iii. no solution.

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 - 4x_2 + 2x_3 &= 7 \\ ax_1 + 11x_2 + 13x_3 &= b \end{aligned}$$

   (c) Use the *Crammer's* rule to solve the following system of linear equations

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 7, \\ x_1 + 2x_2 - 4x_3 &= 3, \\ 3x_1 - 4x_2 - 6x_3 &= 5. \end{aligned}$$