EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2014/2015

FIRST SEMESTER (Nov./Dec., 2016)

PM 201 - VECTOR SPACES AND MATRICES

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wer all questions Time: Three hours

(a) Define what is meant by

(i) a *vector space*;

(ii) a *subspace* of a vector space.

(b) Let $V = \{x : x > 0, x \in \mathbb{R}\}$. Define addition "$\oplus$" and scalar multiplication "$\odot$" as follows:

$$x \oplus y = xy$$

$$r \odot x = x^r$$

for all $r \in \mathbb{R}$ and for all $x, y \in V$. Prove that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

(c) Let $\mathbb{Z}^3$ be the set of tuples of integers with addition '+'and multiplication ' . ' are defined by

$$(l, m, n) + (l', m', n') = (l + l', m + m', n + n'),$$

$$\alpha.(l, m, n) = ([\alpha]\ l, [\alpha]\ m, [\alpha]\ n)$$

where $[\alpha]$ is the integer part of α and $l, m, n, l', m', n' \in \mathbb{Z}$.
Is $(\mathbb{Z}^3, +, .)$ a vector space over the field $\mathbb{R}$? Justify your answer.

2. (a) Let V be a vector space over the field F. Prove the following:

   i. If $v_1, v_2, \cdots, v_m$ are linearly dependent vectors of V such that $v_1, v_2, \cdots, v_{m-1}$ are linearly independent, then $v_m \in \langle\{v_1, v_2, \cdots, v_{m-1}\}\rangle$.

   ii. If u_0 and v_0 are linearly independent vectors of V, and $u_1 = au_0 + bv_0$ and $v_1 = cu_0 + dv_0$, where $a, b, c, d \in F$, then u_1 and v_1 are linearly independent if and only if $ad - bc \neq 0$.

   (b) State the dimension theorem for two subspaces of a finite dimensional vector space.

   Let U_1 and U_2 be subspaces of a vector space V. If $\dim U_1 = 3$, $\dim U_2 = 4$, $\dim V = 6$, show that $U_1 \cap U_2$ contains a non-zero vector.

   If $\dim U_1 = 2$, $\dim U_2 = 4$, $\dim V = 6$, show that $U_1 + U_2 = V$ if and only if $U_1 \cap U_2 = \{0\}$.

   (c) If L is a subspace of a vector space V, prove that there exists a subspace M of V such that $V = L \oplus M$, where $\oplus$ denotes the direct sum.

3. (a) Define:

   (i) *Range space* $R(T)$;

   (ii) *Null space* $N(T)$

   of a linear transformation T from a vector space V into another vector space W.

   Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by:

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z) \text{ for all } (x, y, z) \in \mathbb{R}^3.$$

   Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$ for the linear transformation T.

   (b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x + 2y,\ x + y + z,\ z)$, and let $B_1 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be bases for $\mathbb{R}^3$. Find

(i) The matrix representation of T with respect to the basis B_1;

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrices.

(a) Define the following terms:

(i) *rank* of a matrix;

(ii) *row reduced echelon form* of a matrix.

(b) Let A be an $m \times n$ matrix. Prove the following:

(i) row rank of A is equal to column rank of A;

(ii) if B is a matrix obtained by performing an elementary row operation on A, then A and B have the same rank.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

(a) With the usual notations, prove that

$$A \cdot (adjA) = (adjA) \cdot A = detA \cdot I.$$

Hence, prove that $adj(adj\ A) = (det\ A)^{n-2}A$, where A is a $n \times n$ matrix.

(b) Let J be the $n \times n$ real matrix with every entry equal to 1 and let $A = \alpha I_n + \beta J$, where α, β be real numbers and I_n be the identity matrix of order n.

i. Show that $\det A = \alpha^{n-1}(\alpha + n\beta)$.

ii. If $\alpha \neq 0$ and $\alpha \neq -n\beta$, prove that A is non-singular by finding an inverse for it of the form $\frac{1}{\alpha}(I_n + \gamma J)$.

Determine the inverse of the matrix

$$\begin{bmatrix} 5 & 3 & 3 & 3 & 3 \\ 3 & 5 & 3 & 3 & 3 \\ 3 & 3 & 5 & 3 & 3 \\ 3 & 3 & 3 & 5 & 3 \\ 3 & 3 & 3 & 3 & 5 \end{bmatrix}.$$

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

(b) Find the condition which must be satisfied by y_1, y_2, y_3 and y_4 in order that the system of linear equations

$$\begin{aligned} x_1 - x_3 + 3x_4 + x_5 &= y_1, \\ 2x_1 + x_2 - 2x_4 - x_5 &= y_2, \\ x_1 + 2x_2 + 2x_3 + 4x_5 &= y_3, \\ x_2 + x_3 + 5x_4 + 6x_5 &= y_4 \end{aligned}$$

has solutions.

Find all the solutions for $y_1 = -3, y_2 = 5, y_3 = 6$ and $y_4 = -2$.

(c) State and prove Crammer's rule for 3×3 matrix and use it to solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4; \\ 3x_1 + 5x_2 - x_3 &= -5; \\ -2x_1 - x_2 - 2x_3 &= -5. \end{aligned}$$