EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND YEAR EXAMINATION IN SCIENCE - 2013/2014

SECOND SEMESTER - (OCT., 2016)

PM 204 - RIEMANN INTEGRAL AND SEQUENCES AND SERIES OF FUNCTIONS

(PROPER & REPEAT)

Answer **All** Questions Time Allowed: **2** Hours

Q1. (a) Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function. Suppose that $P := \{x_0, x_1, \ldots, x_n\}$ is a partition of I and ξ_ks are the intermediate points such that $x_{k-1} \leq \xi_k \leq x_k$ for $k = 1, 2, \ldots, n$. Define what is it meant by the *Riemannn sum* of f, $S(P; f; \xi_k)$, corresponding to the partition P and the intermediate points ξ_k.

[10 Marks]

(b) Let $f : I \to \mathbb{R}$ be a continuous function, and let I be divided into n equal sub-intervals of width $\Delta_k := (x_k - x_{k-1}) = (b-a)/n$ for $k = 1, 2, \ldots, n$. If $\xi_k := a + k\Delta_k$, is the right endpoint of the sub-interval $I_k := [x_{k-1}, x_k]$, then show that the definite integral $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S(P; f; \xi_k) = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) \frac{(b-a)}{n}.$$

[30 Marks]

Using the above result, find the value of the definite integral $\int_0^1 (2x+1)\,dx$.

You may use the result $\sum_{k=1}^{n} k = n(n+1)/2$.

[60 Marks]

Q2. (a) State when the integral, $\int_a^b f(x)\,dx$, is said to be an *improper integral* of the

(i) *first kind*;

(ii) *second kind*;

(iii) *third kind*.

[30 Marks]

Determine the convergence of the following integrals:

(i) $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$, where p is a constant and $a > 0$;

(ii) $\displaystyle\int_0^{\infty} \frac{1}{x^3 + x^{1/3}}\,dx$.

[70 Marks]

Q3. (a) Let $\{f_n\}$ be a sequence of functions such that $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, and let $f : A \longrightarrow \mathbb{R}$ be the limit function. Define what it means to say that

(i) the sequence $\{f_n\}$ converges *pointwise* on A to f;

(ii) the sequence $\{f_n\}$ converges *uniformly* on A to f.

[20 Marks]

(b) Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \quad \text{for all } x \in \mathbb{R}.$$

Determine whether $\{f_n\}$ is

(i) pointwise convergent on $\mathbb{R}$.

(ii) uniformily convergent on $\mathbb{R}$.

[80 Marks]

Q4. (a) Let $\{f_n\}$ be a sequence of real-valued functions defined on a subset D of $\mathbb{R}$, and let $f : D \longrightarrow \mathbb{R}$.

(i) Define what is it meant by saying that the infinite series $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on D to f.

(ii) State the *Cauchy Criterion* theorem for the infinite series $\displaystyle\sum_{n=1}^{\infty} f_n$ to be uniformly convergent on D.

[20 Marks]

(b) (i) Prove the *Weierstrass M-test*. Let $\{M_n\}$ be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for all $x \in D$, $n \in \mathbb{N}$, and let the series $\sum_{n=1}^{\infty} M_n$ be convergent, then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D.

[50 Marks]

(ii) Use the Weierstrass M-test or otherwise to show that the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^p}$$

converges uniformly on $\mathbb{R}$ for $p > 1$.

[30 Marks]

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