EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2013/2014

FIRST SEMESTER (May/June, 2016)

PM 302 - COMPLEX ANALYSIS

Proper & Repeat

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nswer all questions **Time: Three hours**

. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.

(b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $z_0 = x_0 + iy_0$, then the derivative $f^{'}(z_0)$ exists.

(c) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being **harmonic**.

Find the harmonic conjugate of $\tan^{-1}\left(\frac{x}{y}\right)$, where $-\pi < \tan^{-1}\left(\frac{x}{y}\right) < \pi$.

2. (a) i. Define what is meant by a **path** $\gamma : [\alpha, \beta] \to \mathbb{C}$.

ii. For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\, dz$.

(b) Let $D(a;r) := \{z \in \mathbb{C} : |z-a| < r\}$ denotes the open disc with center a

radius $r > 0$ and let f be analytic on $D(a;r)$ and $0 < s < r$. Prove C

Integral Formula, that is for $z_0 \in D(a;s)$,

$$f(z_0) = \frac{1}{2\pi i} \int_{C(a;s)} \frac{f(z)}{z - z_0}\, dz,$$

where $C(a;s)$ denotes the circle with center a and radius $s > 0$.

(c) Let $C(0;1)$ be the unit circle $z = e^{it}$, $0 \leqslant t \leqslant 2\pi$. Find the integral

$$\int_{C(0;1)} \frac{dz}{(z-2)(z-\frac{1}{2})}.$$

Hence find that

$$\int_0^{2\pi} \frac{dt}{5 - 4\cos t}.$$

3. (a) State the **Mean-Value Property for Analytic Function.**

(b) i. Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**.

ii. Prove **Liouville's Theorem**: If f is entire and

$$\frac{\max\{|f(t)| : |t| = r\}}{r} \to 0, \quad \text{as} \quad r \to \infty,$$

then f is constant.

(State any results you use without proof).

iii. Prove the **Maximum-Modulus Theorem**: Let f be analytic in an

nected set A. Let γ be a simple closed path that is contained, togeth

inside, in A. Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exist z_0 inside γ such that $|f(z_0)| = M$, then f is constant t

A. Consequently, if f is not constant in A, then

$$|f(z)| < M \quad \text{forall } z \text{ inside } \gamma.$$

4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

   i. f has a pole of order m at z_0;

   ii. f has a zero of order m at z_0.

(b) Show that if f is analytic inside and on a simple closed contour C and f has a pole of order m at $z = \alpha$ then the residue of f at $z = \alpha$ is given by,

$$\lim_{z \to \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-\alpha)^m f(z)\}.$$

If $p(z)$ and $q(z)$ are analytic in the neighborhood of z_0 and $q(z)$ has a zero of order m at z_0 and $p(z)$ has a zero of order $n(\geqslant m)$ at z_0. Show that

$$\lim_{z \to z_0} \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{p^m(z)}{q^m(z)}.$$

(c) Evaluate the integral

$$\int_{C(0;1)} \frac{z^2}{(z^2+1)^2}\, dz.$$

5. Let f be a analytic in the upper-half plane $\{z : \operatorname{Im}(z) \geqslant 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leqslant \frac{M}{|z|^\alpha}, \quad |z| \geqslant R \quad \text{with} \quad \operatorname{Im}(z) \geqslant 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{28 + 11x^2 + x^4}\, dx.$$

(You may assume without proof the Residue Theorem).

6. (a) State the **Principle of Argument Theorem.**

(b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open s
Suppose that

i. f, g are analytic in A except for finitely many poles, none lying on γ.

ii. f and $f+g$ have finitely many zeros in A.

iii. $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denotes the number of zeros - numbe
inside γ of $f+g$ and f respectively, where each is counted as many ti
order.

(c) State the **Fundamental Theorem of Algebra.**

(d) Determine the number of zeros of $p(z) = e^{z^2} - 4z^2$ in the open unit disc D