EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2012/2013)

SECOND SEMESTER (Sept./Oct., 2015)

PM 301 - ALGEBRA III (GROUP THEORY)

(PROPER & REPEAT)

Answer all questions   Time: Three hours

---

1. (a) State what is meant by

   i. an *abelian group*;

   ii. a *cyclic group*.

   (b) Show that the set containing $1, \omega, \omega^2, \cdots, \omega^{n-1}$, the n^{th} roots of unity, forms an abelian group of finite order with usual multiplication. Here $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right), n = 2, 3, \cdots$.

   (c) Let $G = \langle a \rangle$ be a cyclic group of order n and $a \in G$. Prove that

   i. $O(a^m) = \frac{O(a)}{(O(a), m)}$, where $m \in \mathbb{N}$ and $O(a), (O(a), m)$ denote the order of a and the greatest common divisor of $O(a)$ and m, respectively.

   ii. for any $r \in \mathbb{N}$ such that, if r divides n, then G has a subgroup of order r.

   iii. for any $k \in \mathbb{N}$, a^k generates G if and only if $(k, n) = 1$.

   (d) Prove that, for any 2×2 non-singular real matrix and $n \in \mathbb{Z}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

   Hence prove that the set

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

   is a cyclic subgroup of a group of the set of all 2×2 non-singular matrix under the usual matrix multiplication.

2. (a) With the usual notations, prove the following for a group G

   i. $Z(G) = \bigcap_{g \in G} C(g)$;

   ii. $Z(G) \trianglelefteq G$;

   iii. If $G/Z(G)$ is cyclic, then G is abelian.

   (b) State the *Lagrange's* theorem.

   i. Show that every group of prime order is cyclic and that every group of order 4 is abelian.

   Deduce that every group of order less than 6 is abelian.

   ii. Prove that $g^{|G|} = e$ for all $g \in G$, where e is the identity element of G.

   (c) Find the possible order of a non-cyclic subgroup H of a group of order 100 such that H has no elements of order 2.

3. Define the term *isomorphism* as applied to group.

   (a) i. Prove that the composition of two homomorphisms is a homomorphism.

   ii. Prove that homomorphic image of a cyclic group is cyclic.

   (b) State the *First Isomorphism* theorem.

   Let $U_2(\mathbb{Z})$ be the set of 2×2 upper triangular matrices such that

$$U_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}, ad \neq 0 \right\}.$$

   (You may assume, without prove, that $U_2(\mathbb{Z})$ together with matrix multiplication forms a group.)

   Let $\phi : U_2(\mathbb{Z}) \to \mathbb{Z}$ be the mapping defined by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a.$$

   i. Show that ϕ is a homomorphism.

   ii. Find $\ker \phi$.

   iii. Find image of ϕ.

   iv. Prove that $U_2(\mathbb{Z})/\ker\phi \cong \mathbb{Z}$.

4. (a) Let G be a group and $g_1, g_2 \in G$. Define a relation "$\sim$" on G by

   $g_1 \sim g_2$ if and only if there is an element $g \in G$ such that $g_2 = gg_1g^{-1}$.

   Prove that "$\sim$" is an equivalence relation on G.

   Given $a \in G$. Let $\Gamma(a)$ denote the equivalence class containing a.

   If G is a finite group, prove the following:

   i. $|\Gamma(a)| = [G : C(a)]$, where $C(a) = \{x \in G : xa = ax\}$.

   ii. $a \in Z(G)$ if and only if $\Gamma(a) = \{a\}$.

   iii. if $|G| = p^n$, where p is prime and $n \in \mathbb{N}$, then G has non-trivial center.

   (b) Define what is meant by the *internal direct product* as applied to a group.

   Let H and K be two subgroups of a group G. Prove that G is a direct product of H and K if and only if

   i. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

   ii. $hk = kh$ for any $h \in H, k \in K$.

5. (a) Define a *commutator subgroup* G' of a group G.

   Prove the following:

   i. $G' \trianglelefteq G$;

   ii. G/G' is abelian.

   (b) Define the term *p-group*.

   Prove the following:

   i. factor group of a p-group is a p-group.

   ii. homomorphic image of a p-group is a p-group.

   iii. if H and G/H are p-groups, then G is a p-group, provided that H is a normal subgroup of G.

6. (a) Define the following terms as applied to a permutation group.

   i. *cycle of order r*;

   ii. *transposition*;

   iii. *signature*.

(b) Prove that every permutation in S_n can be expressed as a product of transpositions.

(c) Prove that the set of all even permutations, A_n, forms a normal subgroup of S_n, and $|A_n| = \frac{n!}{2}$.
(State any results you may use without proof)

(d) Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 8 & 2 & 1 & 7 & 6 & 3 \end{pmatrix}$ as a product of disjoint cycles. Hence, find out whether it is odd or even.