EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 1994/95 & 95/96(Aug.'97)

REPEAT

MT202 – METRIC SPACE AND RIEMANN INTEGRALS

Time: Two hours.

Answer four questions only.

1. Let (X, ρ) be a metric space and A, B be subsets of X. Define

   (a) the closure $\overline{A}$ of A,

   (b) the interior A^0 of A,

   (c) the distance $\rho(x, A)$ of x from A.

   Prove that

   (a) $(A \cap B)^0 = A^0 \cap B^0$,

   (b) $(A^0 \cup B^0) \subseteq (A \cup B)^0$,

   (c) $\overline{X \setminus A} = X \setminus A^0$,

   (d) $X \setminus \overline{A} = (X \setminus A)^0$,

   (e) $x \in \overline{A}$ if and only if $\rho(x, A) = 0$.

   Illustrate by means of an example that equality does not necessarily hold in part (b).

2. Let (X, ρ) be a metric space and A a subset of X.

   Prove that

   (a) if X is complete and A is closed then A is complete;

   (b) if A is compact then A is closed and bounded;

   (c) every infinite subset of a compact set has a limit point;

   (d) if f is a continuous function on a metric space (X, ρ), then image of a compact set is compact.

3. Let (X, d) be a metric space and $A \subseteq X$. Define the statement that A is connected subset of X.

   (a) Prove that a metric space (X, d) is connected if, and only if the only non empty subset of X, which is both open and closed, is X itself.

   Hence or otherwise prove that if A and B are connected subsets of a metric space (X, d) such that $A \cap B \neq \Phi$, then $A \cup B$ is connected.

   (b) Prove that if $\{A_\alpha / \alpha \in I\}$ is a family of connected sets in X whose intersection is non-empty, then $\bigcup_{\alpha \in I} A_\alpha$ is connected, where I is an indexing set.

4. Let f be a bounded function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With the usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(P, f) - L(P, f) < \epsilon.$$

   (b) Prove that if f is continuous on $[a, b]$, then

      i. f is Riemann integrable over $[a, b]$;

      ii. the function $F : [a, b] \longrightarrow \Re$ defined by $F(x) = \int_a^x f(t)\, dt$ is differentiable on $[a, b]$ and $F'(x) = f(x) \quad \forall x \in [a, b]$;

      iii. if $\phi : [a, b] \longrightarrow \Re$ is positive and Riemann integrable function over $[a, b]$, then there exists a point c in $[a, b]$ such that

      $$\int_a^b f(x)\phi(x)\, dx = f(c) \int_a^b \phi(x)\, dx.$$

5. Prove or disprove each of the following statements. Justify your answers.

   (a) In a metric space the union of a collection of open sets is open.

   (b) If $f : [a, b] \rightarrow \Re$ is define by

   $$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise,} \end{cases}$$

   then f is Riemann integrable over $[a, b]$.

(c) $\int_0^\infty \frac{\cos x}{x^2}\, dx$ converges absolutely.

(d) $\overline{A}$ is the smallest closed set containing A.

6. Discuss the convergence of the following integrals.

(a) $\int_0^1 \frac{e^x}{\sqrt{x}}\, dx$,

(b) $\int_0^\infty \frac{x}{5x^3 + 2x^2 + x + 5}\, dx$,

(c) $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}}\, dx$.