EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 1994/95 & 95/96(Aug.'97)

MT202 – METRIC SPACE AND RIEMANN INTEGRALS

Time: Two hours.

Answer four questions only.

1. Let (X, ρ) be a metric space and A, B be subsets of X. Define

   (a) the closure $\overline{A}$ of A,

   (b) the interior A^0 of A,

   (c) the distance $\rho(x, A)$ of x from A.

   Prove that

   (a) $(A \cap B)^0 = A^0 \cap B^0$,

   (b) $(A^0 \cup B^0) \subseteq (A \cup B)^0$,

   (c) $\overline{X \setminus A} = X \setminus A^0$,

   (d) $X \setminus \overline{A} = (X \setminus A)^0$,

   (e) $x \in \overline{A}$ if and only if $\rho(x, A) = 0$.

   Illustrate by means of an example that equality does not necessarily hold in part (b).

2. Let (X, ρ) be a metric space and A a subset of X.

   Prove that

   (a) if X is complete and A is closed then A is complete;

   (b) if A is compact then A is closed and bounded;

   (c) every infinite subset of a compact set has a limit point;

   (d) if f is a continuous function on a metric space (X, ρ), then image of a compact set is compact.

3. Let (X, d) be a metric space and $A \subseteq X$. Define the statement that A is connected subset of X.

   (a) Prove that a metric space (X, d) is connected if and only if the only non-empty subset of X, which is both open and closed, is X itself.

   Deduce that if A and B are connected subsets of a metric space (X, d) such that $A \cap B \neq \Phi$, then $A \cup B$ is connected.

   (b) Prove that if $\{A_\alpha / \alpha \in I\}$ is a family of connected sets in X whose intersection is non-empty, then $\bigcup_{\alpha \in I} A_\alpha$ is connected, where I is an indexing set.

4. Let f be a bounded function on $[a, b]$. Explain what is meant by the statement that "f **is Riemann integrable over** $[a, b]$".

   (a) With the usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(P, f) - L(P, f) < \epsilon.$$

   (b) Prove that if f is continuous on $[a, b]$, then

   i. f is Riemann integrable over $[a, b]$;

   ii. $$\lim_{n \to \infty} \frac{b - a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b - a)}{n}\right) = \int_a^b f(x)\, dx;$$

   Express $\int_0^1 x^2\, dx$ as a limit of a sum and use the result to evaluate the given definite integral;

   iii. if $\phi : [a, b] \longrightarrow \Re$ is positive and Riemann integrable function over $[a, b]$, then there exists a point c in $[a, b]$ such that

   $$\int_a^b f(x)\phi(x)\, dx = f(c) \int_a^b \phi(x)\, dx.$$

5. Prove or disprove each of the following statements. Justify your answers.

   (a) Every bounded real-valued function f defined on a closed interval $[a, b]$ is Riemann Integrable.

   (b) $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

(c) In a real line $\Re$ with usual metric, the open interval $(0,1)$ is compact.

(d) Let (X,d) be a metric space. If F_1 and F_2 are two disjoint closed subsets of X, then there are two disjoint open sets U and V of X such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

6. (a) Discuss the convergence of the following integrals.

i. $\displaystyle\int_0^1 \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}}\,dx,$

ii. $\displaystyle\int_0^1 \frac{\log x}{1-x^2}\,dx,$

iii. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}}\,dx.$

(b) Show that $\displaystyle\int_0^\infty e^{-x}\sin bx\,dx$ is convergent and determine its value.