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EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Oct./Nov., 2010)

MT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

(PROPER & REPEAT)

Answer all questions Time : Three hours

1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be exact.

Hence solve the following differential equation

$$\left(3x^4y^2 - \frac{1}{y}\right)\frac{dy}{dx} + 4x^3y^3 + \frac{1}{x} = 0.$$

(b) Solve the following differential equations:

i. $\dfrac{2x}{y^3}\,dx + \left(\dfrac{y^3 - 3x^2}{y^4}\right)dy = 0;$

ii. $x\dfrac{dy}{dx} + y = y^2 \ln x;$

iii. $(x + y + 1)\,dx + (2x + 2y + 1)\,dy = 0.$

2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D = \dfrac{d}{dx}$ and p_i, $i = 1, 2, ..., n$, are constants w
$p_0 \neq 0$. Prove the following formulas:

i. $\dfrac{1}{F(D)} e^{\alpha x} = \dfrac{1}{F(\alpha)} e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

ii. $\dfrac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \dfrac{1}{F(D+\alpha)} V$, where V is a function of x.

(b) Find the general solution of the following differential equations by using the resu
in (a).

i. $(D^3 - D)y = e^x + e^{-x}$;

ii. $(D^3 - 3D - 2)y = 540\, x^3 e^{-x}$.

3. (a) Let $x + 1 = e^t$. Show that

$$(x+1)\frac{d}{dx} \equiv \mathcal{D},$$

and

$$(x+1)^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D}.$$

where $\mathcal{D} \equiv \dfrac{d}{dt}$.

Use the above results to find the general solution of the following differential eq
tion

$$[(x+1)^2 D^2 + (x+1)D - 1]y = \ln(x+1)^2 + x - 1.$$

(b) Solve the following simultaneous differential equations:

$$(5D+4)y - (2D+1)z = e^{-x},$$
$$(D+8)y - 3z = 5e^{-x}.$$

4. Use the method of Frobenius to obtain two linearly independent solutions in series for the following differential equation

$$9x^2 y'' + 9x^2 y' + 2y = 0.$$

5. (a) Write down the condition of integrability of the total differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0.$$

Find $f(y)$ such that

$$\left\{\frac{yz + z}{x}\right\} dx - z\,dy + f(y)\,dz = 0$$

is integrable and hence solve the equation.

(b) Solve the following differential equations:

i. $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$;

ii. $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$.

(c) Apply Charpit's method or otherwise to find the complete and the singular solution of the non-linear partial differential equation

$$2z + p^2 = qy + 2y^2 = 0.$$

(d) Solve the non-linear partial differential equation

$$z^2(p^2 + q^2) = 1 - z^2.$$

[You may assume that $z = F(x + ay) = F(u)$, where $u = x + ay$ and a is arbitrary constant.]

6. (a) Obtain the Fourier series expansion of

$$f(x) = \begin{cases} 2x & \text{when } \quad 0 \leqslant x < 3, \\ 0 & \text{when } \quad -3 < x < 0. \end{cases}$$

(b) Find the finite Fourier sine transform and the finite Fourier cosine transform and $\dfrac{\partial^2 u}{\partial x^2}$, where u is a function of x and t for $0 < x < l, \;\; t > 0$.

(c) Use part (b) to show the solution of the partial differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

subject to the boundary condition:

$V(0,t) = 0, \quad V(4,t) = 0, \quad V(x,0) = 2x$, where $0 < x < 4, \; t > 0$, is

$$V(x,t) = \frac{-16}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{-n^2\pi^2 t}{16}} \cos n\pi \sin \frac{n\pi x}{4}.$$