



EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Oct./Nov., 2010)

MT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

(PROPER & REPEAT)

Answer all questions

Time : Three hours

1. (a) State the necessary and sufficient condition for the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

to be exact.

Hence solve the following differential equation

$$\left(3x^4y^2 - \frac{1}{y}\right) \frac{dy}{dx} + 4x^3y^3 + \frac{1}{x} = 0.$$

- (b) Solve the following differential equations:

i. $\frac{2x}{y^3} dx + \left(\frac{y^3 - 3x^2}{y^4}\right) dy = 0;$

ii. $x \frac{dy}{dx} + y = y^2 \ln x;$

iii. $(x + y + 1) dx + (2x + 2y + 1) dy = 0.$

2. (a) If $F(D) = \sum_{i=0}^n p_i D^i$, where $D = \frac{d}{dx}$ and $p_i, i = 1, 2, \dots, n$, are constants with $p_0 \neq 0$. Prove the following formulas:

i. $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

ii. $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D + \alpha)} V$, where V is a function of x .

(b) Find the general solution of the following differential equations by using the results in (a).

i. $(D^3 - D)y = e^x + e^{-x}$;

ii. $(D^3 - 3D - 2)y = 540x^3 e^{-x}$.

3. (a) Let $x + 1 = e^t$. Show that

$$(x + 1) \frac{d}{dx} \equiv \mathcal{D},$$

and

$$(x + 1)^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D}.$$

where $\mathcal{D} \equiv \frac{d}{dt}$.

Use the above results to find the general solution of the following differential equation

$$[(x + 1)^2 \mathcal{D}^2 + (x + 1)\mathcal{D} - 1]y = \ln(x + 1)^2 + x - 1.$$

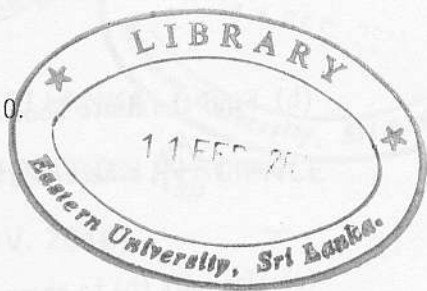
(b) Solve the following simultaneous differential equations:

$$(5D + 4)y - (2D + 1)z = e^{-x},$$

$$(D + 8)y - 3z = 5e^{-x}.$$

4. Use the method of Frobenius to obtain two linearly independent solutions in series for the following differential equation

$$9x^2 y'' + 9x^2 y' + 2y = 0.$$



5. (a) Write down the condition of integrability of the total differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0.$$

Find $f(y)$ such that

$$\left\{ \frac{yz + z}{x} \right\} dx - z dy + f(y) dz = 0$$

is integrable and hence solve the equation.

- (b) Solve the following differential equations:

i. $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2 + y^2};$

ii. $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}.$

- (c) Apply Charpit's method or otherwise to find the complete and the singular solution of the non-linear partial differential equation

$$2z + p^2 = qy + 2y^2 = 0.$$

- (d) Solve the non-linear partial differential equation

$$z^2(p^2 + q^2) = 1 - z^2.$$

[You may assume that $z = F(x + ay) = F(u)$, where $u = x + ay$ and a is arbitrary constant.]

6. (a) Obtain the Fourier series expansion of

$$f(x) = \begin{cases} 2x & \text{when } 0 \leq x < 3, \\ 0 & \text{when } -3 < x < 0. \end{cases}$$

(b) Find the finite Fourier sine transform and the finite Fourier cosine transform and $\frac{\partial^2 u}{\partial x^2}$, where u is a function of x and t for $0 < x < l$, $t > 0$.

(c) Use part (b) to show the solution of the partial differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

subject to the boundary condition:

$V(0, t) = 0$, $V(4, t) = 0$, $V(x, 0) = 2x$, where $0 < x < 4$, $t > 0$, is

$$V(x, t) = \frac{-16}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{-n^2 \pi^2 t}{16}} \cos n\pi \sin \frac{n\pi x}{4}.$$