EASTERN UNIVERSITY, SRI LANKA
FIRST EXAMINATION IN SCIENCE 2005/2006
March/April' 2008
SECOND SEMESTER
MT 102 - REAL ANALYSIS
Proper & Repeat

Answer all questions **Time:Three hours**

Q1. (a) Define the terms *Supremum* and *Infimum* of a bounded subset of $\mathbb{R}$.[10 Marks]

(b) Prove that an upper bound u of a non-empty set S in $\mathbb{R}$ is the supremum of S if and only if for each $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

[25 Marks]

(c) Let A and B be subsets of $\mathbb{R}$ that are bounded, and let

$A + B = \{a + b \ : a \in A, \ b \in A\}$. Prove that

$\sup(A + B) = \sup A + \sup B$. [30 Marks]

(d) Let X be a non-empty set, and let f and g be two functions defined on X and have bounded ranges in $\mathbb{R}$. Show that

$\sup\{f(x) + g(x) \ : \ x \in X\} \leq \sup\{f(x) \ : \ x \in X\} + \sup\{g(x) \ : \ x \in X\}$.

Give examples to show that this inequality can be either equality or strict inequality. [35 Marks]

Q2. (a) Give the formal definition of the notion of a sequence of real numbers converging to a limit. Use this definition to prove that the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.......... converges to a limit, which you should state. [30 Marks]

(b) Use the definition of a limit to show that if x_n and y_n are sequences with $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$ then $x_n + y_n \longrightarrow x + y$ as $n \longrightarrow \infty$. [25 Marks]

(c) Prove that, every convergent sequence of real numbers is bounded. [30 Marks]

(d) Is it true that every bounded sequence of real numbers is convergent? Justify your answer. [15 Marks]

Q3. (a) Let $A \subseteq \mathbb{R}$ with $x_0 \in A$ and let $f : A \longrightarrow \mathbb{R}$ be a function. Define what it means to say that the limit of f at x_0 is ℓ. (ie., $\lim_{x \to x_0} f(x) = \ell$) [10 Marks]

Using the definition, Show that

$$\lim_{x \to 2} \left(\frac{x^3 - 4}{x^2 + 1} \right) = \frac{4}{5} \qquad \text{and} \qquad \lim_{x \to 0} \left(\frac{x^2}{|x|} \right) = 0 \quad (x \neq 0)$$ [40 Marks]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $c \in \mathbb{R}$. Prove that the following conditions are equivalent:

i. $\lim_{x \to c} f(x) = \ell$ exists finitely.

ii. For every sequence (x_n) in $\mathbb{R}$ that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to ℓ. [50 Marks]

Q4. (a) Give the formal definition of what it means for a real-valued function f to be continuous at a point $'a'$ in its domain. [15 Marks]

Prove that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \cos x, \quad \forall\, x \in \mathbb{R}$ is continuous at every point in $\mathbb{R}$. [20 Marks]

(b) Let $I = [a, b]$ be a closed and bounded interval with $a < b$ and let $f : I \longrightarrow \mathbb{R}$ be continuous on I. Prove that f is bounded on I. [35 Marks]

(c) State the *Intermediate Value Theorem* and use it to show that the equation $2x^2(x + 2) - 1 = 0$ has a root in each of the intervals $(-2, -1), (-1, 0)$ and $(0, 1)$. [30 Marks]

Q5. (a) Give the definition of a function $f : [a, b] \longrightarrow \mathbb{R}$ being differentiable at $c \in (a, b)$. Define $f ; [0, \infty) \longrightarrow \mathbb{R}$ as $f(x) = \sqrt{x}$, for $x \geq 0$. Using the definition, show that $f'(c) = \dfrac{1}{2\sqrt{c}}$, for $c > 0$. [35 Marks]

(b) Give an example of a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is not differentiable at $x = 2$. Prove that your function is not differentiable at $x = 2$.

[25 Marks]

(c) State the *Mean Value Theorem.*

Apply the Mean Value Theorem to the function e^x on $[0, b]$, where $0 < b$, to show that

$b < e^b - 1 < be^b$. **[40 Marks]**

Q6. (a) State the *Bolzano-Weierstrass Theorem* for sequence of real numbers. **[15 Marks]**

(b) Give formal definition of what it means for a sequence of real numbers to be a cauchy sequence. **[15 Marks]**

(c) Prove that a sequence of real numbers is convergent if and only if it is a cauchy sequence. **[50 Marks]**

(d) Let (x_n) be a sequence of real number defined by

$x_n = (1 + \frac{1}{2} + ... + \frac{1}{n}) \quad \forall\, n \in \mathbb{N}$. Show that (x_n) is not convergent.

[20 Marks]