EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 2003/2004

(Nov./Dec.'2004)

FIRST SEMESTER

MT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all questions

Time: Three hours

1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} .$$

Hence show that

$$(\underline{a} \wedge \underline{b}) \cdot [(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})] = [\underline{a} \cdot (\underline{b} \wedge \underline{c})]^2 .$$

(b) The vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ are defined, in terms of vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ by

$$\underline{\alpha} = \frac{\underline{b} \wedge \underline{c}}{V} , \underline{\beta} = \frac{\underline{c} \wedge \underline{a}}{V} , \underline{\gamma} = \frac{\underline{a} \wedge \underline{b}}{V} ,$$

where $V = \underline{a} \cdot \underline{b} \wedge \underline{c} \neq 0$.

Show that $\underline{\alpha} \cdot \underline{\beta} \wedge \underline{\gamma} = \frac{1}{V}$. Show also that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{a} + (\underline{r} \cdot \underline{\beta})\underline{b} + (\underline{r} \cdot \underline{\gamma})\underline{c} .$$

(c) Find $\underline{x}$ in terms of $\underline{a}$ and $\underline{b}$ if $\underline{x} \wedge \underline{a} = \underline{b} - \underline{x}$.

2. (a) Define the following terms:

i. the gradient of a scalar field ϕ ;

ii. the divergence of a vector field $\underline{F}$;

iii. the curl of a vector field $\underline{F}$.

(b) Prove the following:

i. $\text{div}(\phi\ \underline{F}) = \phi\ \text{div}\underline{F} + \text{grad}\ \phi \cdot \underline{F}$;

ii. $\text{curl}(\phi\ \underline{F}) = \phi\ \text{curl}\ \underline{F} + \text{grad}\ \phi \wedge \underline{F}$.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$ and let $\underline{a}$ be a constant vector. Evaluate the following:

i. $\text{grad}(\underline{a} \cdot \underline{r})$;

ii. $\text{curl}(\underline{a} \wedge \underline{r})$.

Hence Show that

i. $\text{grad}\left(\frac{\underline{a} \cdot \underline{r}}{r^3}\right) = \frac{\underline{a}}{r^3} - \frac{3\ (\underline{a} \cdot \underline{r})}{r^5}\ \underline{r}$;

ii. $\text{curl}\left(\frac{\underline{a} \wedge \underline{r}}{r^3}\right) = \frac{2\underline{a}}{r^3} + \frac{3\ (\underline{a} \wedge \underline{r})}{r^5} \wedge \underline{r}$.

3. (a) State the Green's theorem in the plane.

Verify the Green's theorem in the plane for

$$\oint_C (xy + y^2)dx + x^2 dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

(b) State the Stoke's theorem.

Verify the Stoke's theorem for a vector

$$\underline{A} = (2x - y)\ \underline{i} - yz^2\ \underline{j} - y^2 z\ \underline{k}$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \quad \text{respectively.}$$

Two particles each of mass m are connected by a light in-extensible string and are lying on a smooth horizontal table with the string taut. The string passes through a small ring O fixed to the table at a point distance $'a'$ from the first particle. The first particle is given horizontal velocity v perpendicular to the string. Prove that it's subsequent path until the second particle reaches O as the polar equation $r = a\sec(\theta/\sqrt{2})$ relative to O. Prove also that if the particle is reached a distance r after time t then $r^2 = a^2 + \frac{1}{2}v^2t^2$.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A smooth wire in the form of an arc of a cycloid which equation is $s = 4a\sin\psi$, is fixed in a vertical plane with the vertex downwards and the tangent at vertex horizontal. A small bead of mass m is threaded on the wire and is projected from the vertex with speed $\sqrt{8ag}$. If the resistance of the medium in which the motion take place is $mv^2/8a$ when the speed is v. Show that the bead comes to instantaneous rest at a cusp $(\psi = \pi/2)$ and returns to the starting point with speed $\sqrt{8ga(1-2e^{-1})}$.

6. State the angular momentum principle.

A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle of mass m is held at the point A on the smooth inner surface of the cone at a distance $'a'$ from the axis of revolution. The particle is projected perpendicular to OA with velocity $'u'$, where O is the vertex of the cone. Show that the particle rises above the level of A if

$$u^2 > ag\cot\alpha$$

and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$