EASTERN UNIVERSITY, SRI LANKA

FIRST YEAR / SECOND SEMESTER EXAMINATION

IN SCIENCE (2002/03 & 2002/03(A))

April/May '2004

MT 102 - ANALYSIS I

Answer **all** questions. Time allowed is **three** hours only. Each questions caries **one hundred** marks. The numbers beside the questions indicate the approximate marks that can be gained from the corresponding parts of the questions.

1. (a) Define the following terms

    i. Supremum
    ii. Infimum

    of a non-empty subset of $\mathbb{R}$. [20]

   (b) State the completeness property of $\mathbb{R}$ and use it to prove the statement that every non-empty bounded below subset of $\mathbb{R}$ has infimum. [30]

   (c) Prove that a lower bound l of a non-empty bounded below subset S of $\mathbb{R}$ is the infimum of S if, and only if for every $\epsilon > 0$, there exists an $x \in S$ such that $x < l + \epsilon$.

    State the corresponding results for supremum. [30]

(d) Let A and B be two non-empty bounded sets of positive real numbers and let C be the set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that
Sup$C$=Sup$A$·Sup$B$. [20]

2. State and prove Archimedean property. [20]

Hence prove the following:

(a) If $x \in \mathbb{R}$, then there exists a unique element $n \in \mathbb{Z}$ such that $n \leq x \leq n+1$. [25]

(b) There exists an element $x \in \mathbb{R}$ such that $x^2 = 2$. [35]

(c) If $x, y \in \mathbb{R}$ with $x < y$, then there exists an element $p \in \mathbb{Q}$ such that $x < p < y$. [20]

3. (a) Explain what is meant by the statement that a sequence of real numbers is

i. convergent,

ii. bounded,

iii. monotone. [30]

(b) Prove that every increasing sequence, which is bounded above is convergent.

Deduce that every decreasing sequence, which is bounded below is convergent. [35]

(c) A sequence of real numbers (a_n) is defined by

$a_1 = x,\ a_{n+1} - {a_n}^2 - y = 0 \quad \forall n \in \mathbb{N}$,

where $0 < y < \dfrac{1}{4}$ and x lies between the roots h and k $(h < k)$ of the equation $a^2 - a + y = 0$.

Prove that

i. $0 < h < a_{n+1} < a_n < k \quad \forall\, n \in \mathbb{N}$

ii. $\lim_{n\to\infty} a_n = h$. [35]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Write out the (ϵ, δ) definition of the statement that f has a limit $l(\in \mathbb{R})$ at a point 'a'$(\in \mathbb{R})$. [10]

(b) Show that if $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} |f(x)| = |l|$.
Is the converse of this result true? Justify your answer. [35]

(c) Show that $\lim_{x\to a} f(x)$ does not exist if and only if there exists a sequence (x_n) in $\mathbb{R}$, which converges to 'a', and $x_n \neq a$ for all $n \in \mathbb{N}$, such that the sequence $(f(x_n))$ does not have a finite limit. [30]

(d) Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that the function g has a finite limit only at $x = 0$. [25]

5. (a) Define what is meant by the statement that $f : \mathbb{R} \to \mathbb{R}$ is

i. continuous on $[a, b]\,(\subseteq \mathbb{R})$

ii. bounded on $[a, b]$. [20]

(b) Let f and g be two continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and suppose that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. [20]

(c) Prove that if a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then it is bounded on $[a, b]$.

Is the converse of this result true?

Give reasons for your answer. [35]

(d) Let $g : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ such that for each $x \in [a, b]$ there exists an element $y \in [a, b]$ such that

$$|f(y)| < \frac{1}{2}|f(x)|.$$

Prove that there exists an element $c \in [a, b]$ such that $f(c) = 0$. [25]

6. (a) Explain what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is

i. differentiable at 'a'$(\in \mathbb{R})$,

ii. strictly increasing at 'a'. [20]

(b) Examine the differentiability of the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

at the origin. [10]

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a real point c and $f'(c) > 0$, then f is strictly increasing at 'c'.

Is the converse of this result true? Justify your answer. [30]

(d) State Mean Value Theorem and use it to prove the following:

If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on $\mathbb{R}$ such that $f(a) = f(b)$, where $a, b\,(\in \mathbb{R})$ are the two consecutive roots of the equation $f''(x) = 0$, then there exists a unique element $c \in (a, b)$ such that f has a local maximum or local minimum at $x = c$.

[40]