EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2005/2006

August/September' 2007

FIRST SEMESTER

MT 203 - EIGEN SPACES AND QUADRATIC FORMS

Answer all questions Time:Two hours

Q1. (a) Define the following terms:

- Skew Symmetric Matrix;
- Orthogonal Matrix.

[10 marks]

Let S is a skew symmetric matrix. Prove that every eigen value of S is the form iy with y real. Deduce that, $(I+S)$ is non singular and prove that $(I-S)(I+S)^{-1}$ is orthogonal. [30 marks]

(b) Let A and B be two matrices in $F_{n\times n}$. Let M be a $2n \times 2n$ matrix of the form $\begin{pmatrix} tI & A \\ B & I \end{pmatrix}$. By pre multiplying M by the matrix $\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}$, prove that $\det M = \psi_{AB}(t)$, where $\psi_{AB}(t)$ is the characteristic polynomial of AB. [15 marks]

(c) Find the eigen values and a basis of each eigen space for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x, y, z) = (-x + 6y - 12z, -13y + 30z, -9y + 20z)$. [30 marks]

(d) Prove that the eigen values of a hermitian matrix (in particular the eigen values of a real symmetric matrix) are real. [15 marks]

Q2. Define the terms "Minimum Polynomial" and "Irreducible Polynomial" of a square matrix.

[10 marks]

(a) State the Cayley Hamilton Theorem.

By evaluating the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 5 & -1 \end{pmatrix},$$

show that $A^{-1} = -\frac{1}{12}(A^2 - 2A - 6I)$, where I is the identity matrix of order 3.

[25 marks]

(b) If $m(t)$ is a minimum polynomial of an $n \times n$ matrix A and $\psi_A(t)$ is the characteristic polynomial of A then $\psi_A(t)$ divides $[m(t)]^n$. [35 marks]

(c) Find the minimum polynomial of the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

[30marks]

Q3. (a) Let λ_1 and λ_2 be two distinct roots of the equation $|A - \lambda B| = 0$, where A and B are real symmetric matrices and let U_1 and U_2 be two vectors satisfying $(A - \lambda_i B)U_i = 0$ for $i = 1, 2$. Prove that $U_1^T B U_2 = 0$ and $U_2^T B U_1 = 0$. [20 marks]

(b) Simultaneously diagonalize the following pair of quadratic form:

$Q_1 = 3x_1^2 + 6x_2^2 + 6x_3^2 + 8x_1x_2 + 8x_1x_3 + 10x_2x_3,$

$Q_2 = 2x_1^2 + 11x_2^2 + 3x_3^2 + 12x_1x_2 + 4x_1x_3 + 14x_2x_3.$

[80 marks]

Q4. (a) Let $x = (x_1, x_2 \ldots\ldots x_n)$, $y = (y_1, y_2, \ldots\ldots y_n) \in \mathbb{R}^n$, where $x_i, y_i \in \mathbb{R}$, $i = 1, 2, \ldots\ldots n$. Let the inner product $< ., . >$ be defined on $\mathbb{R}^n$ as

$$< x, y >= xy^T = \sum_{i=1}^{n} x_i y_i.$$

Show that $(\mathbb{R}^n, < ., . >)$ is an inner product space.

[15 marks]

(b) State and prove the Cauchy-Schwarz Inequality. [20 marks]

(c) State the Gram-Schmidt process.

Find the orthonormal set for span of M in $\mathbb{R}^4$ where

$M = \{(1,0,-1,0)^T, (0,1,2,1)^T, (2,1,-1,0)^T\}$. [45 marks]

(d) If nonzero vectors $\{x_1, x_2 \ldots\ldots x_n\}$ in an inner product space V are mutually orthogonal then prove that they are linearly independent. [20 marks]