

# EASTERN UNIVERSITY, SRI LANAKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS (2001/2002)

(January / February 2004)

MT 406 - APPROXIMATION THEORY

ANSWER ALL QUESTIONS

TIME ALLOWED: 3 HOURS

Q1.

(i) Given that a set  $K$  is convex if  $f, g \in K$  and  $0 \leq \theta \leq 1$  implies  $\theta f + (1 - \theta)g \in K$ . Let  $X$  be a normed space and let  $f \in X$  and  $r > 0$ . Show that the ball  $K := \{g \in X : \|f - g\| \leq r\}$  is convex. [15 marks]

(ii) Let  $X$  be a normed space and let  $K \subseteq X$  be convex. Show that the set of best of approximants to  $f (\in X)$  from  $K$  is a convex set [15 marks]

(iii) Define Uniform Convexity and Strict Convexity of a normed space and state which implies the other. [15 marks]

(iv) Is  $C[-1, 1]$  uniform convex? Justify your answer. [15 marks]

(v) Prove that an (real) inner product space is uniform convex. [20 marks]

(vi) By means of an example, show that the uniform convexity is essential for uniqueness of best approximants [20 marks]

Q2.

(a) Let  $[a, b]$  be given and let  $x_0, x_1, \dots, x_n$  be distinct points in  $[a, b]$ . Let  $f$  be analytic inside and on some simple closed contour  $\Gamma$  containing  $[a, b]$ . Let  $H_n$  be the Hermite interpolation polynomial of degree  $\leq 2n + 1$ , satisfying

$$H_n(x_j) = f(x_j); \quad H'_n(x_j) = f'(x_j), \quad 0 \leq j \leq n$$

and let

$$W(x) := \prod_{j=0}^n (x - x_j).$$

Then show that

$$f(x) = H_n(x) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-x} \left\{ \frac{W(x)}{W(t)} \right\}^2 dt, \quad x \text{ inside } \Gamma.$$

(Hint:  $f - H_n$  has double zeros at  $x_j$ ,  $0 \leq j \leq n$ ) [40 marks]

(b) Define the  $n^{\text{th}}$  Bernstein polynomial  $B_n[f]$ . [5 marks]

(i) Given that  $f_j(x) = x^j$  for  $j = 0, 1, 2$ . Prove that

$$B_n[f_j](x) = f_j(x) \text{ for } j = 0, 1 \text{ and } B_n[f_2](x) = f_2(x) + \frac{1}{n}(x - x^2).$$

[30 marks]

(ii) State Bohman-Korovkin theorem and use it to prove that there exists a sequence  $\{P_n\}_{n=1}^{\infty}$  of polynomials such that  $\lim_{n \rightarrow \infty} P_n = f$  for every  $f \in C[0, 1]$ . Name the result. [25 marks]

Q3.

Let

$$(S_n f)(x) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin\left(\frac{1}{2}\right)t} dt$$

and define the Cesàro means (Fejer Operator) as

$$G_n f := \frac{1}{n} [S_0 f + S_1 f + \dots + S_{n-1} f].$$

(a) Prove that

$$(G_n f)(x) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(t+x) \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^2 dt$$

[35 marks]

(b) Show that the Fejer operators  $G_n$  are monotone linear [20 marks]

(c) Show that  $G_n 1 \rightarrow 1$ ,  $(G_n \cos)(x) \rightarrow \cos x$ , and  $(G_n \sin)(x) \rightarrow \sin x$  as  $n \rightarrow \infty$  and hence show that the Fejer operators of the Fourier series of a continuous  $2\pi$ -periodic function converge uniformly to the function [45 marks]

(Hint: Use the trigonometric analogue of Korovkin's theorem that is given as follows: Let  $\{L_n\}$  denote a sequence of monotone linear operators on  $C_{2\pi}$ . In order that  $L_n f \rightarrow f$  (uniformly) for all  $f \in C_{2\pi}$ , it is necessary and sufficient that such convergence occurs for  $f = 1, \cos$ , and  $\sin$ .)

Q4.

(a) Define a Chebyshev system on a closed interval  $[a, b]$  and show that  $\{e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}\}$  is a Chebyshev system, where  $a_1, a_2, \dots, a_n$  are distinct real numbers. [25 marks]

(b) Chebyshev polynomial of degree  $n$  is defined as

$$T_n(x) := \cos(n \arccos x), \quad x \in [-1, 1], \quad n = 0, 1, 2, \dots$$

(i) Show that  $T_n$  satisfy the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots$$

[25 marks]

(ii) Use (i) above and induction hypothesis to show that  $T_n$  is an algebraic polynomial of degree  $n$  with leading coefficient  $2^{n-1}$ ,  $n \geq 1$  [25 marks]

(iii) Show that  $T_n$  satisfies the following differential equation



$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + n^2y = 0, \text{ where } y = T_n(x).$$

[25 marks]

**Q5.**

(a) If  $g$  is a trigonometric polynomial of degree  $\leq n$  with only cosine terms, show that there exists an ordinary polynomial  $f(x)$  of degree  $\leq n$  such that

$$g(\theta) = f(\cos \theta).$$

[20 marks]

(b) Let  $f \in C[-1, 1]$  and assume that for some  $k \geq 1$ ,  $f^{(k)} \in C[-1, 1]$  and  $g(\theta) = f(\cos \theta)$ ,  $\theta \in [0, 2\pi]$ . Show the following:

(i)

$$\xi_n[g] = E_n[f], \quad n \geq 1;$$

[30 marks]

(ii)

$$E_n[f] \leq \frac{3}{2} \|f'\|_{[-1,1]} \frac{\pi}{n+1};$$

[25 marks]

(iii)

$$\xi_n[g] \leq \left(\frac{3}{2}\right)^{k+1} \pi^k \frac{\omega(f^{(k)}; \frac{\pi}{n-k+1})}{(n+1)n(n-1)\dots(n-k+2)};$$

where  $E_n[f]$ ,  $\xi_n[f]$  denote the errors in approximation of  $f$  by an ordinary and trigonometric polynomial of degree  $\leq n$  respectively and  $\omega(f; \bullet)$  denotes the modulus of continuity. [25 marks]

(You may assume that  $E_n[f] = E_n[f - P]$ , for any ordinary polynomial of degree  $\leq n$ )

**Q6.**

(a) In each of the following three inequalities, give the missing functions (in  $n$ ) or (in  $n$  and  $x$ ) that should appear in the following inequalities. Also name the inequalities. [30 marks]

(i) For trigonometric polynomials of degree  $n$

$$\|R'\|_{[0,2\pi]} \leq \|R\|_{[0,2\pi]}$$

(ii) For ordinary (algebraic) polynomials  $P$  of degree  $n$

$$\|P'\|_{[-1,1]} \leq \|P\|_{[-1,1]}$$

(iii) For ordinary (algebraic) polynomials  $P$  of degree  $n$ , and  $x \in (-1, 1)$

$$|P'(x)| \leq \|P\|_{[-1,1]}$$

4

Give two trigonometric polynomials of degree  $n$  for which there equality in (i) and one algebraic polynomial of degree  $n$  for which there is equality in (ii) above. [15 marks]

(b) Let  $f \in C_{2\pi}$  and  $0 < \alpha < 1$ . Prove that the following are equivalent:

(I) There exists  $A > 0$  such that  $\xi_n[f] \leq An^{-\alpha}$ ,  $n \geq 1$ .

(II) There exists  $B > 0$  such that  $|f(x) - f(y)| \leq B|x - y|^\alpha$ , for  $x, y \in [0, 2\pi]$ .

[30 marks]

(c) Let  $f(\theta) := |\theta|$ ,  $\theta \in [-\pi, \pi]$  and extend  $f$  to  $\mathbb{R}$  by  $2\pi$  periodicity. Prove that there exists  $A > 0$  such that

$$\xi_n[f] \leq An^{-1}, \quad n \geq 1.$$

but that if  $\alpha > 1$ , there does not exist  $A > 1$  such that

$$\xi_n[f] \geq An^{-\alpha}, \quad n \geq 1.$$

[25 marks]

(Here  $\xi_n[f]$  denotes the error in approximation of  $f$  by a trigonometric polynomials of degree  $n$  and  $\mathbb{R}$  denotes the set of real numbers)