EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE

(2002/2003 & 2002/2003(A))

SECOND SEMESTER(Feb./Mar.'2004)

MT 301 - GROUP THEORY

Answer all questions **Time: Three hours**

1. State and prove Lagrange's theorem for a finite group G. [25]

   (a) In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G. [15]

   (b) Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e, \ \forall \ x \in G$. [15]

   (c) Let G be a non-abelian group of order 20. Prove that G contains atleast one element of order 5 or 10. [25]

   (d) Let G be a group of order 27. Prove that G contains a sub group of order 3. [20]

2. (a) State and prove the first isomorphism theorem. [40]

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$; [10]

ii. $H/K \trianglelefteq G/K$; [20]

iii. $\dfrac{G/K}{H/K} \cong G/H$. [30]

3. (a) Define the following terms as applied to a group G.

i. commutator of two elements a, b of G ; [10]

ii. commutator subgroup (G') of G; [10]

iii. internal direct product of two subgroups of G. [10]

(b) Prove that

i. $G' \trianglelefteq G$; [15]

ii. G/G' is abelian. [10]

(c) i. Let H and K be two subgroups of a group G, prove that $G = H \otimes K$ if and only if

A. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

B. $hk = kh$ for any $h \in H, k \in K$. [25]

ii. Give an example to show that a group cannot always be expressed as the internal direct product of two non-trivial normal subgroups. [20]