EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2004/2005

SECOND SEMESTER (Oct./ Nov., 2006)

MT 204 - RIEMANN INTEGRAL & SEQUENCE AND SERIES OF FUNCTIONS

Proper & Repeat

Answer all questions — Time : Two hours

1. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With usual notations, prove that a bounded real valued function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(P, f) - L(P, f) < \epsilon.$$

   (b) With usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for each $\epsilon > 0$ there is a $\delta > 0$ depending on the choice of ϵ such that $|S(P, f, \zeta) - \int_a^b f(x)\,dx| < \epsilon$ for all partition P of $[a, b]$ with $\|P\| < \delta$ and for all selection of the intermediate points ζ.

2. When is an integral $\int_a^b f(x)\,dx$ is said to be improper integral of the first kind, the second kind and the third kind?

What is meant by the statement " an improper integral of the first kind and the second kind are convergent "?

(a) Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$.

(b) Discuss the convergence of the following improper integrals.

i. $\int_1^\infty \frac{\cos x}{x^2}\,dx.$

ii. $\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx.$

iii. $\int_1^\infty \frac{x}{3x^4+5x^2+1}\,dx.$

3. Define the term " uniform convergence of a sequence of functions ".

(a) Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Prove that the sequence (f_n) converges uniformly on A to a bounded function f if and only if for each $\epsilon > 0$ there is a natural number N_ϵ such that for all $m, n \geq N_\epsilon$, then $\|f_m - f_n\| < \epsilon$.

(b) Let f_n be a sequence of functions that are integrable on $[a, b]$ and suppose that (f_n) converges uniformly on $[a, b]$ to f. Prove that f is integrable and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$.

(c) Provide a sequence of functions $\{g_n\}$ converges to a function g on an interval $[0, 1]$ such that $\int_0^1 g_n(x)\,dx$ and $\int_0^1 g(x)\,dx$ exist and $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx$ $\neq \int_0^1 g(x)\,dx$.

4. (a) Let $\{f_n\}$, $\{g_n\}$ be two sequences of functions defined over a non-empty set $E \subseteq \mathbb{R}$. Suppose also that

   i. $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly in E;

   ii. $\sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)| \leq M$ for all $x \in E$, for some $M > 0$;

   iii. $|g_1(x)| \leq M$ for all $x \in E$.

   Prove that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges uniformly in E.

   (b) Prove that $\sum_{k=1}^{\infty} \frac{\sin nx}{n}$ converges uniformly on $[\delta, \pi]$, where $\delta > 0$.