EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2003/2004

June/July 2005

MT 301 - GROUP THEORY

RE-REPEAT

Answer five questions only

Time: Three hours

1. (a) Define the following terms:

   i. Group and,

   ii. subgroup.

   (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall \, a, b \in H$.

   (c) Let H and K be subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

   (d) Let H and K be two subgroups of a group G. Is it true that $H \cup K$ is a subgroup of G? Justify your answer.

   (e) Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G, then prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. (a) State and prove Lagrange's theorem for a finite group G.

(b) In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G.

(c) Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e, \quad \forall x \in G$.

(d) Let G be a non-abelian group of order 10. Prove that G contains at least one element of order 5.

(e) If every non-identity element of a group G has order 2, show that G is abelian.

3. (a) State and prove the first **isomorphism theorem**.

(b) Let H be a subgroup of a group G and K be a normal subgroup of G. Prove with usual notations that,

i. $K \trianglelefteq HK$.

ii. $\dfrac{H}{H \cap K} \cong \dfrac{HK}{K}$.

4. What is meant by " **two elements are conjugate in a group** G "?

(a) Let G be a group and $a, b \in G$. Define a relation " $\sim$ " on G by

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are conjugate in } G.$$

Prove that " $\sim$ " is an equivalence relation on G.

Given $a \in G$, let $\Gamma(a)$ denote the equivalence class containing a.

Show that $|\Gamma(a)| = [G : C(a)]$, and $a \in Z(G) \Leftrightarrow \Gamma(a) = \{a\}$, where $C(a) = \{x \in G \,/ ax = xa\}$ and $Z(G)$ is the center of the group G.

(b) Write down the class equation of a finite group G.

Hence or otherwise, prove that if the order of G is p^n, where p is a prime number and n is a positive integer then the center of G is non-trivial.

5. (a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. G' is a normal subgroup of G.

iii. $\frac{G}{G'}$ is abelian.

6. Prove or disprove the following:

(a) Let G be a group and $Z(G)$ be the centre of G. If $\frac{G}{Z(G)}$ is cyclic then G is abelian.

(b) Let G be a finite group. Then $O(ab) = O(ba)$ for all $a, b \in G$. ($O(x)$ stands for the order of the element x.)

(c) Every abelian group is cyclic.

(d) Let $\Phi : G \to G_1$ be a homomorphism, where G and G_1 are two groups. If H is a normal subgroup of G then $\Phi(H)$ is a normal subgroup of G_1.

(e) Homomorphic image of a p-group is p-group.

7. Define the following terms:

* homomorphism

* isomorphism

* automorphism and inner automorphism.

(a) Prove the following:

i. homomorphic image of an abelian group is abelian.

ii. homomorphic image of a cyclic group is cyclic.

(b) Let ***AutG*** be the set of all automorphism of a group G and let ***InnG*** be the set of all inner automorphism of G. Show that,

i. ***AutG*** is a group under composition of maps.

ii. ***InnG*** is a normal subgroup of ***AutG***.

8. (a) Define the following terms in the symmetric group S_n $(n \geq 2)$:

i. transposition.

ii. cycle of order r $(1 \leq r \leq n)$.

iii. signature of a permutation.

Using the first isomorphism theorem or otherwise prove that the set of all even permutations of S_n forms a normal subgroup of S_n.

(b) Express the permutation f in S_8 as a product of disjoint cycles, where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}.$$