EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2003/2004

SECOND SEMESTER (Apr.'2006)

MT 301 - GROUP THEORY

REPEAT

Answer all questions

Time allowed: Three hours

1. (a) Define the following terms:

    i. group and,

    ii. subgroup of a group.

   (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall\, a, b \in H$.

   (c) Let H and K be subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

   (d) Let H and K be two subgroups of a group G. Is it true that $H \cup K$ is a subgroup of G? Justify your answer.

   (e) Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G. Prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. State and prove **Lagrange's theorem** for a finite group G.

(a) Let H and K be two different subgroups of a group G with order p, where p is prime. Prove that $H \cap K = \{e\}$, where e is the identity element of G.

(b) Let x and y be elements of a finite group G. Show that the element $x^{-1}yx$ has the same order as y.

(c) If every non-identity element of a group G has order 2, show that G is abelian.

3. (a) State and prove the **first isomorphism theorem** .

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

4. Prove or disprove the following:

(a) Let G be a group and $Z(G)$ be the centre of G. If $\dfrac{G}{Z(G)}$ is cyclic then G is abelian.

(b) If G is a finite group then $O(ab) = O(ba)$ for all $a, b \in G$.
($O(x)$ stands for the order of the element x.)

(c) Every abelian group is cyclic.

(d) Let $\Phi : G \to G_1$ be a homomorphism, where G and G_1 are two groups. If H is a normal subgroup of G then $\Phi(H)$ is a normal subgroup of G_1.

(e) Homomorphic image of a p-group is p-group.

5. (a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. G' is a normal subgroup of G.

iii. $\frac{G}{G'}$ is abelian.

6. Define the following terms:

* homomorphism

* isomorphism

* automorphism and inner automorphism.

(a) Prove the following:

i. homomorphic image of an abelian group is abelian.

ii. homomorphic image of a cyclic group is cyclic.

(b) Let $AutG$ be the set of all automorphisms of a group G and let $InnG$ be the set of all inner automorphisms of G. Show that,

i. $AutG$ is a group under composition of maps.

ii. $InnG$ is a normal subgroup of $AutG$.