

EASTERN UNIVERSITY, SRI LANKA
THIRD EXAMINATION IN SCIENCE, (2005/2006)
(December, 2006)
FIRST SEMESTER
SPECIAL REPEAT
MT 302 - COMPLEX ANALYSIS

Answer all questions

Time allowed: 3 Hours

- Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \rightarrow \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$. [20]
- (b) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \rightarrow \mathbb{C}$ be differentiable at some $z_0 = x_0 + i y_0 \in A$. If $f(z) = u(x, y) + i v(x, y)$, then prove that the partial derivatives of $u(x, y)$ and $v(x, y)$ satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $z_0 = x_0 + i y_0$.

[50]

- (c) Find the set of complex numbers at which the function

$$f(x + iy) = 2xy + i \left(x + \frac{2}{3}y^3 \right)$$

is differentiable.

[30]

- Q2. (a) (i) Define what is meant by a **path** $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}$. [10]
 (ii) For a path γ and a continuous function $f : \gamma \rightarrow \mathbb{C}$, define $\int_{\gamma} f(z) dz$. [10]

(b) Let $a \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$. Show that

$$\int_{C(a; r)} (z - a)^n dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1 \end{cases}$$

where $C(a; r)$ denotes a positively oriented circle with centre a and radius r . [30]

(State but **do not prove** any results you may assume).

(c) State the **Cauchy's Integral Formula**. [20]

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\int_{C(0;3)} \frac{e^{zt}}{z^2 + 1} dz, \quad t > 0;$ [15]

(ii) $\int_{C(0;3)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z - 1)(z - 2)} dz$ [15]

where $C(0; 3)$ denotes a positively oriented circle with centre 0 and radius 3.

Q3. (a) State the **Mean Value Property for Analytic Functions**. [10]

(b) (i) Define what is meant by the function $f : \mathbb{C} \rightarrow \mathbb{C}$ being **entire**. [10]

(ii) Prove **Liouville's Theorem**: If f is entire and

$$\frac{\max \{|f(t)| : |t| = r\}}{r} \rightarrow 0, \quad \text{as } r \rightarrow \infty,$$

then f is constant. [30]

(State any results you use without proof)

(c) Prove the **Maximum-Modulus Theorem**: Let f be analytic in an open connected set A . Let γ be a simple closed path that is contained, together with its inside, in A . Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A . Consequently, if f is not constant in A , then

$$|f(z)| < M \quad \forall z_0 \text{ inside } \gamma.$$

[50]

(State any theorem you use without proof)

Q4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \rightarrow \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

- (i) f having a singularity at z_0 ;
- (ii) the order of f at z_0 ;
- (iii) f having a pole or zero at z_0 of order m ;
- (iv) f having a simple pole or simple zero at z_0 .

[40]

(b) Prove that

$$\text{ord}(f; z_0) = m$$

if and only if

$$f(z) = (z - z_0)^m g(z), \quad z \in D^*(z_0; \delta),$$

for some $\delta > 0$, where g is analytic in $D(z_0; \delta)$ and $g(z_0) \neq 0$.

[60]

Q5. (a) Prove that if f has a simple pole at z_0 , then

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z).$$

[30]

(b) Let f be analytic in $\{z : \text{Im}(z) \geq 0\}$, except possibly for finitely many singularities, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^\alpha}, \quad |z| \geq R \text{ with } \text{Im}(z) \geq 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x) dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

[50]

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx.$$

[20]

(You may assume without proof the Residue Theorem).

Q6. (a) State the **Principle of the Argument Theorem**. [20]

(b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open starset A . Suppose that

(i) f, g are analytic in A except for finitely many poles, none lying on γ .

(ii) f and $f + g$ have finitely many zeros in A .

(iii) $|g(z)| < |f(z)|$, $z \in \gamma$. Then

$$ZP(f + g; \gamma) = ZP(f; \gamma)$$

where $ZP(f + g; \gamma)$ and $ZP(f; \gamma)$ denote the number of zeros – number of poles inside γ of $f + g$ and f respectively, where each is counted as many times as its order. [40]

(c) State the **Fundamental theorem of Algebra**. [20]

(d) Prove that the equation $2e^z + z + 3 = 0$ has exactly one root in the left-half plane. [20]