EASTERN UNIVERSITY, SRI LANKA
SECOND EXAMINATION IN SCIENCE - 2005/2006
SECOND SEMESTER (March/April, 2008)
MT 204 - RIEMANN INTEGRAL AND SEQUENCES AND SERIES OF FUNCTIONS
(PROPER AND REPEAT)

Answer all Questions | Time: Two hours

Q1. (a) State and proof the necessary and sufficient condition for the integrability of a bounded function on $[a, b]$. Hence show that if f is Riemann integrable and there exists a number c such that $L(P, f) < c < U(P, f)$ if and only if

$$|U(P, f) - c| < \varepsilon \text{ and } |c - L(P, f)| < \varepsilon.$$ [60 marks]

(b) Prove or disprove the followings:

(i) Every continuous function is Riemann integrable;

(ii) $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q}^c \end{cases}$ is Riemann integrable. [40 marks]

Q2. (a) Explain an improper integral of the first, second and third kinds. [30 marks]

(b) Test the convergence of the following using an appropriate test:

(i) $\int_0^\infty \frac{1}{x^p}\, dx$;

(ii) $\int_0^\infty \frac{\cos x}{1+x^2}\, dx$. [50 marks]

(c) Find the values of m and n for which the following integral converges.

$$\int_0^1 e^{-mx} x^n\, dx.$$ [20 marks]

Q3. (a) What is meant by uniform convergence of a sequence of function $\{f_n\}_{n\in\mathbb{N}}$

Show that the sequence $\{f_n\}$ defined by

$$f_n(x) = x^n$$

on [0,1] converges for every $x \in [0,1]$. Does it converge uniformly? Justify y

answer.

[35 ma

(b) Comment on the following statement with suitable example.

"Every pointwise convergent sequence of functions is uniformly converge

[30 ma

(c) Read the following statement carefully.

Suppose $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a metric space X

$\lim\limits_{n\to\infty} f_n(x) = f(x)\ \forall x \in X$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f if

only if $M_n \longrightarrow 0$ as $n \longrightarrow \infty$, where $M_n = \sup\{|f_n(x) - f(x)| : x \in X$

Using the above result, show that the sequence $\{f_n\}_{n\in\mathbb{N}}$, where

$$f_n(x) = nx(1-x)^n$$

does not converge uniformly on [0,1].

[35 ma

Q4. (a) What is meant by a series of real-valued functions converges uniformly

$E \subseteq \mathbb{R}$.

[10ma

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on $\mathbb{E} \subseteq \mathbb{R}$. Supp

that for each $n \in \mathbb{N}$, there is a constant M_n such that $|f_n(x)| \leq M_n$; $\forall x$

where $\sum\limits_{n=1}^{\infty} M_n$ converges. Prove that $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on E.

Hence, show that the series

$$\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

converges uniformly on $\mathbb{R}$.

[40ma

(c) State Abel's test and show that the series of functions

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^3}$$

uniformly converges on $\mathbb{R}$.

[50ma