

EASTERN UNIVERSITY, SRI LANKA THIRD EXAMINATION IN SCIENCE - 2005/2006 SECOND SEMESTER (Sep.'2007) <u>MT 301 - GROUP THEORY</u>

REPEAT

Answer all questions

Time allowed: Three hours

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Lanka

Sri

IBRARI

- 1. (a) Define the following terms:
 - i. group,
 - ii. subgroup of a group.
 - (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if ab⁻¹ ∈ H, ∀ a, b ∈ H.
 - (c) Let H and K be subgroups of a group G. Prove that HK is a subgroup of G if and only if HK = KH.
 - (d) Let H and K be two subgroups of a group G. Is it true that $H \cup K$ a subgroup of G? Justify your answer.
 - (e) Let $\{H_{\alpha}\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G. Prove that $\bigcap_{\alpha \in I} H_{\alpha}$ a subgroup of G.

- 2. (a) State and prove Lagrange's theorem for a finite group G.
 - (b) Prove that in a finite group G, the order of each element divides order of G. prove that every group of prime order is cyclic, moreover, in a group of prime p, every non-identity element has order p.
 - (c) Let p and q be two distinct prime numbers and let G be a group of order pq. that every proper subgroup of G is cyclic.
 - (d) Let G be a non-cyclic group of order 8. Show that $a^4 = e$ for every $a \in G$, we is the identity element of G.
- 3. (a) State and prove the first isomorphism theorem .
 - (b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove
 - i. $K \trianglelefteq H$; ii. $H/K \trianglelefteq G/K$; iii. $\frac{G/K}{H/K} \cong G/H$.
- 4. Prove or disprove the following:
 - (a) Let G be a group and Z(G) be the centre of G. If G/Z(G) is cyclic then G is abe
 - (b) If G is a finite group then O(ab) = O(ba) for all a, b ∈ G.
 (O(x) stands for the order of the element x.)
 - (c) Every abelian group is cyclic.
 - (d) Let $\Phi: G \to G_1$ be a homomorphism, where G and G_1 are two groups. If H normal subgroup of G then $\Phi(H)$ is a normal subgroup of G_1 .
 - (e) Homomorphic image of a *p*-group is *p*-group.

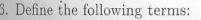
(a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

- ii. G' is a normal subgroup of G.
- iii. G/G' is abelian.



- * homomorphism;
- * isomorphism;
- * automorphism and inner automorphism.
- (a) Prove the following:
 - i. homomorphic image of an abelian group is abelian.
 - ii. homomorphic image of a cyclic group is cyclic.
- (b) Let AutG be the set of all automorphisms of a group G and let InnG be the set of all inner automorphisms of G. Show that,
 - i. AutG is a group under composition of maps.
 - ii. InnG is a normal subgroup of AutG.

