EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

(2004/2005)

MARCH/APRIL' 2007

PART II

MT 40[illegible] - FUNCTIONAL ANALYSIS

Answer all questions **Time: Three hours**

1. a) Let $\{x_1, x_2, x_3, \ldots \ldots, x_n\}$ be a linearly independent set of vectors in a normed linear space X. Prove that there is a number c>0 such that for every choice of scalars $\alpha_1, \alpha_2, \ldots \ldots, \alpha_n$

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \ldots \ldots + \alpha_n x_n \| \geq c\,(|\alpha_1| + |\alpha_2| + \ldots \ldots + |\alpha_n|).$$

[60 Marks]

b) Prove that every finite dimensional subspace Y of a normed linear space is complete.

[40 Marks]

2. a) Let X, Y and Z be normed linear spaces. Prove that a linear operator $T: X \to Y$ is continuous if and only if there is a number $K \geq 0$ such that

$$\| Tx \| \leq K \; \| x \| \quad (x \in X).$$

[40 Marks]

b) Define the norm, $\| T \|$ of a bounded linear operator $T: X \to Y$.

[10 Marks]

c) Let B[X,Y] be the vector space of all bounded linear operators of X into Y. Show that $\| \; \|$ as defined in (b) is a norm on B[X,Y]. (You may assume that B[X,Y] is a vector space.)

[25 Marks]

d) Let $S \in B[X, Y]$ and $T \in B[Y, Z]$. Let T o S be the linear operator defined by

$$(T \circ S)(x) = T(S(x)) \qquad (x \in X).$$

Show that T o S is a bounded linear operator with

$$\| T \circ S \| \leq \| T \| \; \| S \|.$$

[15 Marks]

e) If $T \in B[X,Y]$ and $S \in B[Y,X]$ are such that $T \circ S$ is the identity operator in Y, what relation can you deduce between $\| T \|$ and $\| S \|$?

[10 Marks]

3. a) State the Hahn Banach Theorem for (real and complex) normed linear spaces.

[15 Marks]

b) Prove the Hahn Banach Theorem for complex normed spaces, assuming that it holds for real normed spaces.

[55 Marks]

c) Let X be a normed linear space. Use the Hahn Banach theorem to show that for every $x \in X$,

$$\| x \| = \text{Sup} \{ | f(x) | : f \in X^*, \| f \| \leq 1 \}$$

[30 Marks]

4. a) Let X,Y be normed linear spaces, and (T_n) be a sequence of bounded linear operators of X into Y. What does it mean to say that (T_n) is

- uniformly bounded;
- point wise bounded?

[15 Marks]

b) State and prove the Uniform Boundedness Theorem on the relation between two concepts defined in part(a) under suitable conditions. (You may assume the Baire's Category theorem.)

[45 Marks]

c) Let X = Y = Coo , the space of all eventually zero sequences $x = (x_1, x_2, x_3, \ldots)$ with the supremum norm

$$\| x \| = \text{Sup} \{ |x_i| : i = 1, 2, 3, \ldots \}.$$

Show that the sequence of linear operators (T_n) defined by

$$T_n(x) = \{ x_1, 2x_2, 3x_3, \ldots, nx_n, 0, 0, \ldots \} \quad (x \in \text{Coo})$$

are pointwise bounded but not uniformly bounded. Why does this not contradict the Uniform Boundedness theorem?

[45 Marks]

5. **a)** State the Open Mapping theorem.

[15 Marks]

b) Let T be bounded linear operator from a Banach space X onto a Banach space Y. Prove that T has the property that the image T(Bo) of the open unit ball Bo = $B(0,1) \subset X$ contains an open ball about $0 \in Y$. (You may assume the Baire's Category theorem)

[85 Marks]

6. **a)** Let H be a Hilbert space. Prove that every bounded linear functional f on H can be represented in terms of the inner product namely,

$$f(x) = \langle x, z \rangle \quad (x \in H.)$$

where $z \in H$ depends on f. Further show that z is uniquely determined by f and has the norm $\|z\| = \|f\|$.

[60 Marks]

b) Define what is meant by a normed linear space is separable.
Prove with the usual notations that the sequence space l^p with $1 \leq p < \infty$ is separable.

[40 Marks]