EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - (2007/2008)

FIRST SEMESTER

(December/January, 2008/2009)

MT 302 - COMPLEX ANALYSIS

PROPER & REPEAT

Answer **all** questions — Time allowed: **3 Hours**

Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$. [20]

(b) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be differentiable at some $z_0 = x_0 + i\,y_0 \in A$. If $f(z) = u(x, y) + i\,v(x, y)$, then prove that the partial derivatives of $u(x, y)$ and $v(x, y)$ satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

at $z_0 = x_0 + i\,y_0$. [50]

(c) (i) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being harmonic. [10]

(ii) Suppose that the function $F(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Show that the functions $u(x, y)$ and $v(x, y)$ are harmonic in D. [20]

Q2. (a) (i) Define what is meant by a **path** $\gamma : [\alpha, \beta] \to \mathbb{C}$. [10]

(ii) For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\,dz$. [10]

(b) Let $a \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$. Show that

$$\int_{C(a;\, r)} (z-a)^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1 \end{cases}$$

where $C(a; r)$ denotes a positively oriented circle with centre a and radius r. [30]

(State any results you use without proof).

(c) State the **Cauchy's Integral Formula**. [20]

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\displaystyle\int_{C(0;\,2)} \frac{e^z}{\pi i - 2z}\,dz;$ [15]

(ii) $\displaystyle\int_{C(0;\,2)} \frac{z^3}{z^2 - 2z + 3}\,dz.$ [15]

Q3. (a) State the **Mean Value Property for Analytic Functions**. [10]

(b) (i) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**. [10]

(ii) Prove **Liouville's Theorem**: If f is entire and bounded then f is constant. [30]

(State any results you use without proof).

Suppose that the function $J(z) = u(x, y) + iv(x, y)$ is analytic everywhere in the xy-plane and $u(x, y)$ bounded for all (x, y) in the xy-plane. Prove that $u(x, y)$ is constant throughout the plane. [10]

(c) Prove the **Maximum-Modulus Theorm**: Let f be analytic in an open connected set A. Let γ be a simple closed path that is contained, together with its inside, in A. Let

$$M := \sup_{z\in\gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A. Consequently, if f is not constant in A, then

$$|f(z)| < M \quad \forall\, z \text{ inside } \gamma.$$

[40]

(State any theorem you use without proof).

Q4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

(i) f having a singularity at z_0;

(ii) the order of f at z_0;

(iii) f having a pole or zero at z_0 of order m;

(iv) f having a simple pole or simple zero at z_0. [40]

(b) Prove that

$$\text{ord}(f; z_0) = m$$

if and only if

$$f(z) = (z - z_0)^m\, g(z), \quad \forall\, z \in D^*(z_0; \delta),$$

for some $\delta > 0$, where g is analytic in $D(z_0; \delta) := \{z : |z - z_0| < \delta\}$ and $g(z_0) \neq 0$.

[40]

(c) Prove that if f has a simple pole at z_0, then

$$\text{Res}(f; z_0) = \lim_{z\to z_0} (z - z_0) f(z_0).$$

[20]

Q5. Let f be analytic in $\{z : \mathrm{Im}(z) \geq 0\}$, except possibly for finitely many singularities, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^{\alpha}}, \quad |z| \geq R \text{ with } \mathrm{Im}(z) \geq 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

[60]

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx.$$

[40]

(You may assume without proof the Residue Theoerem).

Q6. (a) State the **Argument Theorem**. [20]

(b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open starset A. Suppose that

(i) f, g are analytic in A except for finitely many poles, none lying on γ.

(ii) f and $f+g$ have finitely many zeros in A.

(iii) $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denote the number of zeros – number of poles inside γ of $f+g$ and f respectively, where each is counted as many times as its order. [40]

(c) State the **Fundamental theorem of Algebra**. [20]

(d) Prove that the equation $R(z) = z^5 + 3z^3 + 6 = 0$ has exactly one root in the left-half plane. [20]