EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION IN MATHEMATICS (2004/2005)
MARCH/APRIL, 2007
PART I
MT 412 – FUNCTIONS OF SEVERAL VARIABLES AND APPLICATIONS

Time: 3 Hours *Maximum Marks: 600*

Answer ALL Questions

I. (*a*) Suppose $f = (f_1, \ldots, f_m): D \to R^m$ and that a is a limit of D and $b = (b_1, \ldots, b_m) \in R^m$. Prove that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f_i(x) = b_i$, $i = 1, \ldots, m$.

(*b*) Let $f: A \subset R^n \to R^m$ and let x_0 be in A or a boundary point of A. Show that $\lim_{x \to x_0} f(x) = b$ if and only if, for every number $\varepsilon > 0$, there is a $\delta > 0$ such that, for $x \in A$ satisfying $0 < \| x - x_0 \| < \delta$, we have $\| f(x) - b \| < \varepsilon$.

(*c*) Let $U \subset R^n$ and $V \subset R^m$ be open. Let $g: U \subset R^n \to R^m$ and $f: V \subset R^m \to R^p$ be given functions such that g maps U into V so that $f \circ g$ is defined. Suppose g is differentiable at x_0 and f at $y_0 = g(x_0)$. Then prove that $f \circ g$ is differentiable at x_0 and that $D(f \circ g)(x_0) = Df(y_0)\, Dg(x_0)$.

(*d*) Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$

Is f differentiable at (0,0)? Prove your assertion. [*20 + 20 + 30 + 30* = **100**]

II. (*a*) If $f: R^n \to R^m$ is differentiable at x_0, prove that the directional derivative $D_r f(x_0)$ exists for all $r \in R^n$ and $D_r f(x_0) = d f_{x_0}(h)$.

(*b*) Prove that if f is continuously differentiable at x_0, then f is differentiable at x_0.

(*c*) Let $f: R^3 \to R^4$ be defined by $f(x_1, x_2) = (x_2, x_1, x_1 x_2, x_2^2 - x_1^2)$. Let $a = (1,2)$. Determine the tangent plane to the image S of f at the point f(a).

(*d*) Let f be a real-valued function, defined on the open set U in R^n. If the first and second partial derivatives of f exist and are continuous in U, prove that $D_i D_j f = D_j D_i f$ on U.

[*15 + 25 + 40 + 20* = **100**]

III. (*a*) Let $f: U \subset R^n \to R$ have continuous partial derivatives of third order. Show that

$$f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \tfrac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(h, x_0) \text{ where}$$

$R_2(h, x_0) / \| h \|^2 \to 0$ as $h \to 0$.

(*b*) The graph of the function $g(x,y) = 1/xy$ is a surface S in R^2. find the points of S that are closest to the origin (0,0,0).

(*c*) Find the rectangle box with volume 1000 having the least total surface area. [*25 + 35 + 40* = **100**]

IV. (*a*) Let f: $U \subset R^n \to R^m$ be a C^1 mapping where U is a neighborhood of the line segment L with end points a and b. Prove that $|f(b) - f(a)|_0 \le |b - a|_0 \max_{x \in L} \|f'(x)\|$.

(*b*) Suppose that the mapping f: $R^n \to R^n$ is C^1 in a neighborhood W of the point a, with the matrix f'(a) being nonsingular. Prove that f is locally invertible – i.e., there exist neighborhoods $U \subset W$ of a and V of b = f(a), and a one-to-one C^1 mapping g: $V \to W$ such that g(f(x)) = x for $x \in U$ and f(g(y)) = y for $y \in V$; and, in particular, prove that the local inverse g is the limit of the sequence $\{g_k\}_0^\infty$ of successive approximations, defined inductively by

$g_0(y) = a$, $g_{k+1}(y) = g_k(y) - f'(a)^{-1}[f(g_k(y)) - y]$ for $y \in V$.

(*c*) Let the C^1 mapping f: $R^2_{uv} \to R^2_{xy}$ be defined by the equations

$$x = u + (v + 2)^2 + 1$$
$$y = (u - 1)^2 + v + 1.$$

Let a = (1,–2). Is f invertible near a? If so, find a local inverse of f. [*25* + *35* + *40* = 100]

V. (*a*) State the General Implicit Mapping Theorem.

Solve
$$x^2 + \tfrac{1}{2} y^2 + z^3 - z^2 - 3/2 = 0$$
$$x^3 + y^3 - 3y + z + 3 = 0$$
for y and z as functions of x in a neighborhood of (-1,1,0).

(*b*) Prove that every admissible function is integrable.

(*c*) Let f: $R^{m+n} = R^m \times R^n \to R$ be an integrable function such that, for each $x \in R^m$, the function f_x: $R^n \to R$, defined by $f_x(y) = f(x,y)$, is integrable. Given the contented sets $A \subset R^m$ and $B \subset R^n$, let F: $R^m \to R$ be defined by $F(x) = \int_B f_x = \int_B f(x,y)dy$. Then prove that F is integrable, and

$$\int_{A \times B} f = \int_A F = \int_A \left(\int_B f(x,y)dy \right) dx.$$

(*d*) Find the mass of the ellipsoidal ball $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ with the uniform density of unity.

[*30* + *20* + *30* + *20* = 100]

VI. (*a*) If f is a real-valued C^1 function on the open set $U \subset R^n$ and γ:[a,b] $\to$ U is a C^1 path, prove that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

(*b*) If α is a C^1 differential k-form on an open subset of R^n, prove that $d(d\alpha) = 0$.

(c) If φ: $R^m \to R^n$ is a C^1 mapping and α is a C^1 differential k-form, show that $d(\varphi^*\alpha) = \varphi^*(d\alpha)$.

(*d*) Let Q = [0,1] × [0,1] $\subset R^2$ and suppose φ: Q $\to R^3$ is defined by the equations

$$x = u + v,$$
$$y = u - v,$$
$$z = uv.$$

Then compute the surface integral $\int_\varphi x\,dy \wedge dz + y\,dx \wedge dz = \int_\varphi \alpha$ in two different methods you are aware of. [*20* + *20* + *30* +*30* = 100]