EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS-(2003/2004)

(JUL./AUG.' 2005)

Part II

MT 405 - FUNCTIONAL ANALYSIS

Answer all questions

Time: Three hours

1. (a) Let Y and Z be subspaces of a normed linear space X, and suppose that Y is closed and is a proper subset of Z. Prove that for every real number θ in the interval $(0,1)$ there is a $z \in Z$ such that

$$\| z \| = 1 \quad \text{and} \quad \| z - y \| \geq \theta \quad \text{for all } y \in Y.$$

[40 marks]

(b) Prove that, if a normed linear space X has the property that the closed unit ball $B = \{ x \in X \mid \| x \| \leq 1 \}$ is compact then X is finite dimensional. **[30 marks]**

(c) Let X and Y be normed linear spaces. An operator $T : X \to Y$ is called **compact** linear operator if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact, that is $\overline{T(M)}$ is compact.

i. Prove that, every compact linear operator $T : X \to Y$ is bounded. **[15 marks]**

ii. Is it true that every bounded linear operator $T : X \to Y$ is compact? Justify your answer. **[15 marks]**

2. (a) The linear space $B(X, Y)$ of all bounded linear operators from a normed linear space X into a normed linear space Y is itself a normed linear space with norm defined by

$$\| T \| = \sup_{\substack{x \in X \\ \| x \| = 1}} \| Tx \| .$$

Prove that, if Y is a **Banach** space then $B(X, Y)$ is a Banach space. [40 marks]

(b) Define the term **"dual space X^* of a normed linear space X"** [10 marks]

Prove with the usual notations that, the dual space of l^1 is l^∞.

[50marks]

3. (a) State the Baire **"category"** theorem and use it to prove the **Uniform boundedness** theorem. [50 marks]

(b) Let X be a normed linear space and E be a subset of X. Prove that E is bounded in X if and only if, $f(E)$ is bounded for every $f \in X^*$. [50 marks]

4. Prove or disprove the following:

(a) The sequence space l^∞ with the usual norm is separable.

[40 marks]

(b) In a normed linear space **weak convergence** implies **strong convergence**. [30 marks]

(c) Let X and Y be normed linear spaces and let (T_n) be a sequence of bounded linear operators in $B[X, Y]$.

If $T : X \to Y$ is such that

$$\| T_n x - Tx \|_Y \longrightarrow 0 \quad \text{for all} \quad x \in X \quad \text{as } n \longrightarrow \infty,$$

then

$$\| T_n - T \|_{B[X,Y]} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

[30 marks]

5. (a) Define a **closed** linear operator from a normed linear space X into a normed linear space Y. [15 marks]

   (b) State the **Open Mapping** Theorem and used it to prove the **Closed Graph** Theorem. [50 marks]

   (c) Give an example of a closed linear operator which is not bounded. Justify your answer. [35 marks]

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6. (a) Let M be a closed subspace of a Hilbert space H and let $x \in H$. Show that there exists a unique $x_0 \in M$ such that

$$\| x - x_0 \| = \inf_{m \in M} \| x - m \| \quad \text{and} \quad x - x_0 \perp M.$$

[50 marks]

   (b) State the **Hahn-Banach** theorem for a normed linear space X.

[10 marks]

   i. Let X be a normed linear space and $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional ϕ on X such that

$$\| \phi \| = 1 \quad \text{and} \quad \phi(x_0) = \| x_0 \|.$$

[20 marks]

   ii. If $f(x) = f(y)$ for every bounded linear functional f on a normed linear space X, show that $x = y$. [20 marks]