EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

(2003/2004 - July/August, 2005)

Part II

MT 411 - NUMERICAL THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Answer all questions

Time: Three hours

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1. (a) Let (α_n) be a sequence of real numbers satisfying

$$\alpha_{n+1} \leq (1+A)\,\alpha_n + B\,, \quad n = 0, 1, 2, \cdots,$$

with $A > 0$ and $B \geq 0$. Prove that

$$\alpha_n \leq (1+A)^n\,\alpha_0 + \Big[(1+A)^n - 1\Big]\frac{B}{A}\,, \quad n = 0, 1, 2, \cdots.$$

(b) Let $y \in C^2[0,1]$ be the solution of the m-dimensional system

$$y' = f(x,y), \quad 0 \leq x \leq 1, \quad y(0) = \nu,$$

where, for the maximum norm

$$\| f(x,u) - f(x,v) \| \;\leq\; L \, \| u - v \|$$

with $L > 0$ for all $u, v \in \mathbb{R}^m$ and for all $x \in [0,1]$.

Show that, for any x and h,

$$\| y(x+h) - y(x) - h\,y'(x) \| \;\leq\; \frac{1}{2}\,h^2 \cdot \max_{0 \leq x \leq 1} \| y''(x) \| .$$

(c) For given y_0, let $y_1, y_2, \cdots, y_N$ be given by the explicit Euler method

$$y_{n+1} = y_n + hf(nh, y_n), \quad n = 0, 1, 2, \cdots, N-1$$

with $Nh = 1$.

Show that

$$\| y(1) - y_N \| \leq e^L \| y_0 - \nu \| + \frac{h}{2L} (e^L - 1) \max_{0 \leq x \leq 1} \| y''(x) \|$$

and comment briefly on the significance of this result.

2. (a) Define the following terms.

i. Consistency,

ii. Zero-stability,

iii. Convergence,

applied to the linear multi-step method

$$\sum_{j=0}^{k} \alpha_j \, y_{j+1} = h \sum_{j=0}^{k} \beta_j \, f_{n+j} \,, \qquad (\alpha_k = 1),$$

used for solving initial-value problems of the form

$$y' = f(x, y), \;\; x \in [a, b], \;\; y(a) = \eta,$$

where $f : [a, b] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$.

(b) Prove that a convergent linear multi-step method is consistent. (Any result used here should be stated.)

(c) Show that for all values of α, the method

$$y_{n+2} + \alpha \, y_{n+1} - (1 + \alpha) \, y_n = \frac{h}{2} \left[-\alpha f_{n+2} + (4 + 3\alpha) \, f_{n+1} \right]$$

is consistent. For which values of α is the method convergent?

3. (a) Define the order of a linear multi-step method in terms of the associated linear operator.

Show that the linear multi-step method

$$y_{n+2} - 2\, y_{n+1} + y_n = \frac{h}{2}\left(f_{n+2} - f_n\right)$$

is of order three.

(b) Show that a linear multi-step method with characteristic polynomials ρ and σ is of order p if and only if

$$\rho(z) - \ln(z)\sigma(z) = C_{p+1}(z-1)^{p+1} + C_{p+2}(z-1)^{p+2} + \cdots ,$$

$|z-1| < 1$ with $C_{p+1} \neq 0$.

(c) A linear multi-step method with characteristic polynomial

$$\rho(z) = z^2 - z$$

is of maximum order. Determine the method and the error constant and explain why the method is convergent.

4. (a) Define the term "absolute stability" (at a point $q \in \mathbb{C}$) as applied to a numerical method for solving initial-value problems for ordinary differential equations.

(b) State a necessary and sufficient condition for a linear multi-step method, with characteristic polynomials ρ and σ , to be absolutely stable for given $q \in \mathbb{C}$, in terms of the roots of the polynomial $\rho(r) - q\,\sigma(r)$.

Show that a convergent linear multi-step method cannot be absolutely stable for small real positive q.

(c) Consider the two-step method

$$y_{n+2} - \frac{4}{3}\, y_{n+1} + \frac{1}{3}\, y_n = \frac{2}{3}\, h\, f_{n+2}.$$

Write down $\rho(r)$ and $\sigma(r)$ associated with this method and show that if r is a root of the polynomial $p(r) - q\sigma(r)$, then

$$q = \frac{3}{2} - 2\, r^{-1} + \frac{1}{2}\, r^{-2}.$$

Hence show that if $q \in \partial S$ (where ∂S denotes the boundary of the region of absolute stability), then

$$q(\theta) = (\cos\theta - 1)^2 + i\ \sin\theta\ (2 - \cos\theta)$$

for some $\theta \in [0, 2\pi]$. Find the interval of absolute stability of the method.

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5. (a) An s-stage Runge-Kutta method with coefficients array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}, \qquad c_i = \sum_{j=1}^{n} a_{ij}\ , \quad i = 1, 2, \cdots, s,$$

is used to solve initial-value problems of the form

$$y' = f(y), \ \ y(a) = \nu, \ \ f : \mathbb{R}^m \to \mathbb{R}^m.$$

Write down the formulae that define y_{n+1}, approximation to $y(x_n + h)$, given an approximation y_n to $y(x_n)$.

(b) Define the order of a Runge-Kutta method.

Prove that all explicit s-stage Runge-Kutta methods of order s have identical regions of absolute stability. Determine the interval of absolute stability when $s = 2$.

(c) Prove that the three-stage Runge-Kutta method coefficients

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

is of order 2 and that the interval of absolute stability is $[-4, 0]$.
[The order conditions may be assumed but should be stated].

6. (a) The coefficients of an s-stage Runge-Kutta method are given by the array

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}, \quad C = Ae, \quad e = [1, 1, \cdots, 1]^T, \; b^T = [b_1, b_2, \cdots, b_s].$$

Let $B = \text{diag}(b_1, b_2, \cdots, b_s)$. Define the terms "B-stability" and "algebraic stability" as applied to the Runge-Kutta method.

(b) Prove that, if a Runge-Kutta method is algebraically stable, then it is B-stable.

(c) Is the two-stage Runge-Kutta method given by the array

$$\begin{array}{c|cc} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{4} & \frac{3}{8} & \frac{3}{8} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

algebraically stable? Justify your answers.