EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

ACADEMIC YEAR - 2008/2009 (December, 2010)

Part II

MT 405 - NUMERICAL THEORY OF ORDINARY DIFFERENTIAL EQUATION

Answer all questions. **Time allowed: Three hours**

1. (a) Let (x_n) be a sequence of real numbers satisfying

$$x_{n+1} \leq (1+\lambda)x_n + \lambda\mu, \; n = 0, 1, 2, \ldots, \qquad \text{where } \lambda > 0 \text{ and } \mu \geq 0.$$

Prove that

$$x_n \leq e^{n\lambda}x_0 + (e^{n\lambda} - 1)\mu, \quad n = 0, 1, 2, \ldots.$$

(b) Let $y \in C^2[0,1]$ be the solution of m-dimensional system

$$y' = f(x, y), \quad 0 \leq x \leq 1, \quad y(0) = \nu,$$

where for some norm

$$\| f(x,u) - f(x,v) \| \; \leq \; L\|u - v\| \; \text{ and } \; \|y''(x)\| \leq K$$

with $L > 0$ for all $u, v \in \mathbb{R}^m$ and for all $x \in [0,1]$.

Show that for any x and h,

$$\| y(x+h) - y(x) - hy'(x)\| \; \leq \; \frac{1}{2} h^2 K.$$

(c) For given y_0, let $y_1, \; y_2, \; \ldots, \; y_N$ be given by the explicit Euler method

$$y_{n+1} = y_n + hf(nh, y_n), \quad n = 0, 1, 2, \ldots, N-1,$$

where h is chosen so that $Nh = 1$.

Show that

$$\|y(1) - y_N\| \; \leq \; e^L\|\nu - y_0\| + \frac{hK}{2L}(e^L - 1).$$

2. (a) Define the terms **Convergence**, **Consistency** and **Zero-stability** of a linear multi-step method

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k} \beta_j \, f_{n+j}, \quad (\alpha_k = 1),$$

used for solving initial value problems of the form

$$y' = f(x, y), \; a \le x \le b, \; y(a) = \nu,$$

where

$$y : [a, b] \to \mathbb{R}^m \text{ and } f : [a, b] \times \mathbb{R}^m \to \mathbb{R}^m.$$

(b) State **Dahlquist's Theorem** giving necessary and sufficient conditions for a linear multi-step method to be convergent.

(c) Write down the first and second characteristic polynomials, $\rho(x)$ and $\sigma(x)$ of the linear multi-step method and show that the method is consistent with the initial value problem if and only if

$$\rho(1) = 0 \text{ and } \rho'(1) = \sigma(1).$$

(d) Find the values of the parameter α for which the linear multi-step method

$$y_{n+2} - (1-\alpha)y_{n+1} - \alpha y_n = h[(1+2\alpha)^2 f_{n+2} - 2(1+\alpha)f_{n+1} + (1-\alpha)f_n]$$

is convergent

3. (a) Define the **Order** of the linear multi-step method in terms of the associated linear operator. Show that the order of the method

$$y_{n+2} - (1+\alpha)y_{n+1} + \alpha y_n = \frac{1}{2}h(1-\alpha)(f_{n+1} + f_n)$$

is independent of the parameter α.

(b) Show that a linear multi-step method with characteristic polynomials ρ and σ is of order p if and only if

$$\rho(z) - \ln(z)\sigma(z) = C_{p+1}(z-1)^{p+1} + C_{p+2}(z-1)^{p+2} + \dots,$$

$|z - 1| < 1$ with $C_{p+1} \neq 0$.

(c) A linear multi-step method with characteristic polynomial

$$\rho(z) = z^2 - z - 1$$

is of maximum order. Find the method and the error constant.

4. (a) Define the **Stability polynomial** of a linear multi-step method in terms of the characteristic polynomials associated with the method.

   (b) State a necessary and sufficient condition for a linear multi-step method, to be absolute stable for given $z \in \mathbb{C}$, in terms of the roots of the stability polynomial.

   (c) Show that the expression for the locus of ∂R_A (where ∂R_A denotes the boundary of the region of absolute stability) for the two step method

$$y_{n+2} - y_{n+1} = \frac{h}{2}(3y_{n+1} - y_n)$$

   is given by

$$z(\theta) = \frac{2(2\cos\theta - \cos^2\theta - 1)}{5 - 3\cos\theta} + i\frac{2\sin\theta(2 - \cos\theta)}{5 - 3\cos\theta}$$

   for some $\theta \in [0,\ 2\pi]$.

   Deduce the interval of absolute stability; check your result by using Routh -Hurwitz criterion to find the interval of absolute stability.

5. (a) The coefficient of an s-stage Runge-Kutta method are given by the Butcher array

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}, \qquad C = Ae, \quad e = (1,\ 1,\ 1, \ldots,\ 1)^T.$$

   Show that the method is absolutely stable for given $z \in \mathbb{C}$ if det $(I - zA) \neq 0$ and $|R(z)| \leq 1$, where $R(z) = 1 + zb^T(I - zA)^{-1}e$.

   Deduce that, for an s-stage explicit method, $R(z)$ is a polynomial of degree s and hence, prove that all explicit s-stage Runge-Kutta methods of order s have identical region of absolute stability.

   (b) Determine the interval of absolute stability of the two-stage explicit Runge-Kutta method represented by the Butcher array

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

   This method is used to solve the initial-value problem

$$y' = -(2x+1)y^2, \quad y(0) = 1$$

   on $[0,\ 1]$. Find an appropriate bound for the step-length so that the choice of step-length does not conflict with the criterion for absolute stability.

6. (a) An implicit Runge-Kutta method for the first order initial-value problem

$$y' = f(x, y), \quad y(a) = \nu$$

is given by

$$\begin{aligned} k_1 &= hf(x_n + \frac{h}{3},\, y_n + \frac{5}{12}k_1 - \frac{1}{12}k_2), \\ k_2 &= hf(x_n + h,\, y_n + \frac{3}{4}k_1 - \frac{1}{4}k_2), \\ y_{n+1} &= y_n + \frac{3}{4}k_1 + \frac{1}{4}k_2, \quad n = 0,\ 1,\ 2,\ \ldots, \end{aligned}$$

where h is the step-length, $x_n = a + nh$. When this method is applied to the differential equation $y' = \lambda y$, show that

$$\left(1 - \frac{2}{3}\lambda h + \frac{1}{6}\lambda^2 h^2\right) y_{n+1} = \left(1 + \frac{1}{3}\lambda h\right) y_n.$$

Hence, show that the implicit Runge-Kutta method is absolutely stable for all $\lambda h < 0$.

(b) Prove that the one parameter family of semi-implicit methods given by the array

$\frac{3\mu-1}{6\mu}$	$\frac{3\mu-1}{6\mu}$	0
$\frac{1+\mu}{2}$	μ	$\frac{1-\mu}{2}$
	$\frac{3\mu^2}{3\mu^2+1}$	$\frac{1}{3\mu^2+1}$

is algebraically stable for all $\mu \le 0$.