EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION IN MATHEMATICS
(2001/2002) PART II (Jan./Feb.2004)
MT 309-NUMERICAL LINEAR ALGEBRA

You may attempt as many questions as you wish, but marks will be given for the best **FOUR** answers only. Time allowed is **THREE** hours only. Each question carries **ONE HUNDRED** marks. The numbers beside the questions indicate the approximate marks that can be gained from the corresponding parts of the questions.

1. (a) Define the term "positive definite" as applied to an $n \times n$ Hermitian matrix A. Prove that the determinant of each principal sub-matrix of a positive definite matrix is positive. [20]

   (b) Define the term "elementary lower triangular matrix". Prove that a positive definite matrix can be uniquely expressed as $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix. [30]

   (c) Show that a Hermitian matrix A is positive definite if, and only if $A = GG^H$, where G is a non-singular lower triangular matrix. [25]

   Determine G such that

$$GG^H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 4 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}.$$

[25]

2. (a) Define the terms "unitary matrix" and "elementary Hermitian matrix ". [10]

(b) Show that, for any real vector x, there is a real elementary Hermitian matrix $H(w)$ such that $H(w)x = ce_1$, where $c = x^T x$ and $e_1 = (1, 0, 0, \ldots, 0)^T$. What is the optimal choice of the sign of c for the computation of w? [20]

(c) Let $H(w)$ be an $n \times n$ elementary Hermitian matrix and let I be the $m \times m$ identity matrix. Show that the partitioned matrix

$$\left[\begin{array}{c|c} I & O \\ \hline O & H(w) \end{array}\right]$$

is an elementary Hermitian matrix. [20]

(d) Determine an upper triangular matrix U such that $HA = U$, where H is a product of elementary Hermitian matrices and

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 2 & 5 & 0 \end{bmatrix}$$

making the optimal choice of sign in each stage of process. Hence solve the system $Ax = b$, where $b = (5, 0, -1)^T$ [50]

3. (a) Define the phrase strictly diagonally dominant applied to an $n \times n$ matrix A. [05]

(b) Given that A is strictly diagonally dominant, prove that A is non-singular. Given also that $A = I - L - U$, where L is strictly lower triangular and U is strictly upper triangular. Verify that $\| L + U \|_\infty < 1$ and obtain a bound for $\| A^{-1} \|_\infty$. [25]

(c) For arbitrary $x^{(0)}$, a sequence $\{x^{(r)}\}$ is defined by

$$x^{(r+1)} = (I - wL)^{-1}\{wb + [(1 - w)I + wU]x^{(r)}\}, \quad r = 0, 1, 2, 3 \ldots$$

Show that $x - x^{(r+1)} = M(x - x^{(r)})$, $r = 0, 1, 2, \ldots$, where $M = (I - wL)^{-1}[(1-w)I + wU]$ and $Ax = b$. State a necessary and sufficient condition for $\{x^{(r)}\}$ to converge to x. [10]

(d) Let $0 < w \leq 1$ and let λ be any complex number with $|\lambda| \geq 1$. Show that $|\lambda + w - 1| \geq |w\lambda| \geq w$. Deduce that if λ is any eigenvalue of M, then $|\lambda| < 1$. [30]

(e) The following equations are to be solved by successive over-relaxation with a relaxation parameter 1.1.
Starting with $x^{(0)} = 0$, obtain $x^{(1)}$, $x^{(2)}$ and bound for $\| x - x^{(2)} \|_\infty$.

$$\begin{bmatrix} 11 & 1 & 0 & 0 \\ 1 & 11 & 1 & 0 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 2 & 11 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

[30]

4. (a) Define the term "upper Hessenberg matrix." [05]

(b) i. Let A be an $n \times n$ matrix. Describe how a non-singular matrix S, a product of elementary lower triangular matrices and elementary permutation matrices, can be obtained so that $S^{-1}AS$ is an upper Hessenberg matrix. [30]

ii. Obtain the number of multiplications needed for this process. Explain why Householder's method is better than this process when A is Hermition. [30]

(c) Given

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

find an upper Hessenberg matrix $S^{-1}AS$, where S is a product of elementary permutation matrices and elementary lower triangular matrices. [35]

5. (a) Let $A = (a_{ij})$ be an $n \times n$ upper Hessenberg matrix such that $a_{21}a_{32}\ldots a_{nn-1} \neq 0$.

Show that the characteristic polynomial $p(\lambda)$ of A is given by

$$p(\lambda) = \alpha_{n+1}(\lambda)a_{21}a_{32}\ldots a_{n\ n-1},$$

where α_{n+1} is given by the recurrence relation

$$\alpha_r\lambda = \alpha_1 a_{1r} + \alpha_2 a_{2r} + \cdots + \alpha_r a_{rr} + \alpha_{r+1}a_{r+1r},\ r = 1, 2, \ldots, n,$$

α_r is a function of λ, $r = 1, 2, \ldots, n$.

(Assume that $\alpha_1 = 1$, $a_{n+1\ n} = 1$).

[35]

(b) The upper Hessenberg matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has one eigenvalue λ in $(3.4, 3.5)$. Find the characteristic polynomial of A, and its derivative at $\lambda_0 = 3.5$. Apply one step of the Newton method to obtain a new estimate for λ. [65]

6. (a) i. Suppose that the eigenvalue λ_1 of largest modulus and corresponding eigenvector z_1 of an $n \times n$ matrix A have been computed by the Power method. Show that there is a non-singular matrix S, a product of an elementary permutation matrix and an elementary lower triangular matrix, such that

$$A = S \left[\begin{array}{c|c} \lambda_1 & \gamma^T \\ \hline O & B \end{array} \right] S^{-1},$$

where B is an $(n-1) \times (n-1)$ matrix and γ is an $(n-1)$-column vector.

[25]

ii. Describe how the other eigenvalues and eigenvectors of A could be computed.

[20]

(b) It is given that the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

has an eigenvalue close to 3.4 and that a corresponding eigenvector approximately $(0.7, 1, 0.3)^T$. Obtain 2×2 matrix B whose eigenvalues approximate the other eigenvalues of A .

[30]

(c) Describe briefly how inverse iteration is used to improve an approximate eigenvalue and eigenvector of an $n \times n$ matrix.

[25]