EASTERN UNIVERSITY, SRILANKA
DEPARTMENT OF MATHEMATICS
FIRST EXAMINATION IN SCIENCE – 2007/2008
SECOND SEMESTER (August/September, 2009)
ST 104 – DISTRIBUTION THEORY
(REPEAT)

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er all questions   Time : Three hours

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f X and Y are two random variables having joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & if\ 0<x<2, 2<y<4 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) marginal densities of X and Y.

(b) joint cumulative distribution function.

(c) $P(X<1, Y<3)$.

(d) $P(X+Y<3)$.

(e) $P(X<1|Y<3)$.

(a) A particular fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and leaving the service window, and Y_2, the time that a customer waits in line before reaching the service window. The relative frequency distribution of observed values of Y_1 and Y_2 is modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}; & 0 \le y_2 \le y_1 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

The random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window.

i. Find the probability density function for U.

ii. Find $E(U)$ and $V(U)$

(b) If X is a random variable with mean μ and variance σ^2, then for any positive number k, prove that

$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$.

3. (a) Suppose that the length of time Y that takes a worker to complete a certain task, has the probability density function

$$f(y) = \begin{cases} e^{-(y-\theta)}; & y > \theta, \\ 0 & \text{otherwise.} \end{cases}$$

where θ is a positive constant that represents the minimum time to task completion. Let $Y_1, Y_2, ..., Y_n$ denote a random sample of completion times from this distribution.

i. Find the density function for $Y_{(1)} = min(Y_1, ..., Y_n)$.

ii. Find $E(Y_{(1)})$.

(b) Let X be a standard normal variate. Show that $Y = X^2$ is a chi-square random variable with degrees of freedom 1.

(c) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a normal distribution with a mean μ and a variance of σ^2. If $Z_i = \dfrac{(Y_i - \mu)}{\sigma}$, show that $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left[\dfrac{(Y_i - \mu)}{\sigma}\right]^2$ is a χ^2 distribution with n degrees of freedom.

4. (a) The joint density function of X and Y is given by

$$f_{XY}(x, y) = \frac{1}{2\Pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y\ldots}{}\right.\right.\right.$$

where $x, y, \mu_1, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0$ and $|\rho| \leq 1$

i. Find the marginal density function of X. Name the density function.

ii. Find the conditional density function of Y given $X = x$. Name the density function.

iii. From (ii) deduce $E(Y|X = x)$.

) A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1.0$ ounce. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottles are with the same machine setting) and the ounce of fill measured for each. Find the probability that the sample mean differ from the true mean within 0.3 ounce for that particular setting.

t Y_1 and Y_2 have the joint density function given by

$$f(y_1, y_2) = \begin{cases} K y_1 y_2; & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the value of K that makes this a probability density function.

b) Find the joint cumulative distribution function for Y_1 and Y_2 .

(c) Find $P(Y_1 \le \frac{1}{2},\ Y_2 \le \frac{3}{4})$.

A certain process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and Y_2 the proportion of type I impurity among all impurities found. Suppose the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function:

$$f(y_1, y_2) = \begin{cases} 2(1 - Y_1); & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the probability density function of the proportion of type I impurities in the sample.

(b) Find the expected value of the proportion of type I impurities in the sample.