EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

2001/2002 (Jan.'2004)(PART II)

MT 403 - MEASURE THEORY

Answer four questions only

Time : Three hours

1. Explain what is meant by a step-function on $\mathbb{R}$.

   (a) Let (ϕ_n) be an increasing sequence of step functions on $\mathbb{R}$ such that $\left(\int \phi_n\right)$ is convergent. Prove from first principle that (ϕ_n) converges almost everywhere on $\mathbb{R}$.

   (b) Show that a subset E of $\mathbb{R}$ is null if, and only if, there is an increasing sequence (ϕ_n) of step functions on $\mathbb{R}$ such that

   i. $\lim_{n\to\infty} \int \phi_n < \infty$ and

   ii. for all $x \in \mathbb{E}$, $\quad \phi_n(x) \to \infty$ as $n \to \infty$.

2. (a) Let $f \in L^1(\mathbb{R})$. Prove that there exists a sequence (ϕ_n) of step functions such that $\phi_n(x) \to f(x)$ almost everywhere in $\mathbb{R}$, and that

$$\int |f - \phi_n| \to 0 \quad \text{as} \quad n \to \infty.$$

(b) Prove that the function $x \to f(x) \cos kx$ belongs to $L^1(\mathbb{R})$ for each $k \in \mathbb{R}$, and that

$$\lim_{k \to \infty} \int_{\mathbb{R}} f(x) \cos kx \, dx = 0.$$

(c) Given examples of sequences (g_n) and (h_n) in $L^1(\mathbb{R})$ such that

i. $g_n \to 0$ almost everywhere but $\int g_n \nrightarrow 0$ and

ii. $\int h_n \to 0$ but $h_n \nrightarrow 0$ almost everywhere.

3. (a) Let f be a function which vanishes outside the interval $[a, b]$. Prove that if f is bounded and if the points of discontinuity of f form a null set then $f \in L^1(\mathbb{R})$.

(b) State the Monotone convergence Theorem in $L^1(\mathbb{R})$ and use it to prove

i. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|} \, dx = 2,$

ii. If $f \in L^1$ then $\int |f| = 0$ if, and only if, $f = 0$ almost everywhere, and

iii. $\displaystyle\lim_{n \to \infty} \int_0^1 \frac{1+x}{1+x^n} \, dx = \frac{3}{2}.$

4. State and prove the Dominated convergence theorem in $L^1(\mathbb{R})$.

(Monotone convergence theorem may be assumed.)

(a) Prove that, for $\beta > -1$,

$$\int_0^\infty \frac{e^{-\beta x}}{e^x + e^{-x}}\, dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1+\beta)}.$$

(b) Let $p,\ q$ be real numbers such that $0 < p < q$. Prove that

$$\int_0^1 \frac{x^p - x^q}{x(1-x)}\, dx = (q-p)\sum_{n=0}^{\infty} \frac{1}{(n+p)(n+q)}.$$

5. State Fubini's Theorem for Lebesgue integrable function on $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be measurable and suppose that one of the repeated integrals

$$\int \left[\int |f(x,y)|\, dx \right] dy, \quad \int [\, |f(x,y)|\, dy]\, dx$$

exists. Prove that $f \in L^1(\mathbb{R}^2)$.

(a) Prove that

$$f(x,y) = (x - \sin x) y^2 e^{-xy} \in L^1\left((0,\infty)\times(0,\infty)\right)$$

and deduce that

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx = \frac{\pi}{4}.$$

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x,y) &= \frac{x^2 - y^2}{(x^2+y^2)^2} \quad &\text{if } (x,y) \neq (0,0) \\ &= 0 \quad &\text{if } (x,y) = (0,0). \end{aligned}$$

Does $f \in L^1((0,1)\times(0,1))$? Justify your answer.

6. (a) Define the term measurable function.

    i. Prove that, $f : \mathbb{R} \to \mathbb{R}$ is measurable if, and only if $\text{mid}(-g, f, g)$ is integrable whenever g is a non-negative integrable function.

    ii. Let (f_n) be a sequence of measurable functions on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ almost everywhere. Prove that f is measurable.

    iii. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $c \in \mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid f(x) \leq c\}$ is measurable.

    (State without proof, any convergence theorem that you use.)

   (b) What is meant by "a set $A \subseteq \mathbb{R}$ is null"?

   Prove that an interval in $\mathbb{R}$ is null if, and only if, it is degenerate.