EASTERN UNIVERSITY, SRI LANAKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS (2001/2002)

(January / February 2004)

MT 406 - APPROXIMATION THEORY

ANSWER ALL QUESTIONS **TIME ALLOWED: 3 HOURS**

Q1.

(*i*) Given that a set K is convex if $f, g \in K$ and $0 \leq \theta \leq 1$ implies $\theta f + (1-\theta)g \in K$. Let X be a normed space and let $f \in X$ and $r > 0$. Show that the ball $K := \{g \in X : \|f - g\| \leq r\}$ is convex. [15 marks]

(*ii*) Let X be a normed space and let $K \subseteq X$ be convex. Show that the set of best of approximants to $f (\in X)$ from K is a convex set [15 marks]

(*iii*) Define Uniform Convexity and Strict Convexity of a normed space and state which implies the other. [15 marks]

(*iv*) Is $C[-1,1]$ uniformy convex? Justify your answer. [15 marks]

(*v*) Prove that an (real) inner product space is uniformy convex. [20 marks]

(*vi*) By means of an example, show that the uniform convexity is essential for uniqueness of best approximants [20 marks]

Q2.

(*a*) Let $[a,b]$ be given and let $x_0, x_1, \ldots x_n$ be distinct points in $[a,b]$. Let f be analytic inside and on some simple closed contour Γ containing $[a,b]$. Let H_n be the Hermite interpolation polynomial of degree $\leq 2n+1$, satisfying

$$H_n(x_j) = f(x_j); \quad H_n'(x_j) = f'(x_j), \qquad 0 \leq j \leq n$$

and let

$$W(x) := \prod_{j=0}^{n} (x - x_j).$$

Then show that

$$f(x) = H_n(x) + \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-x} \left\{ \frac{W(x)}{W(t)} \right\}^2 dt, \ \ x \text{ inside } \Gamma.$$

(Hint: $f - H_n$ has double zeros at x_j, $0 \leq j \leq n$) [40 marks]

(*b*) Define the n^{th} Bernstein polynomial $B_n[f]$. [5 marks]

(*i*) Given that $f_j(x) = x^j$ for $j = 0, 1, 2$. Prove that

$$B_n[f_j](x) = f_j(x) \ \text{ for } j = 0, 1 \text{ and } B_n\,[f_2](x) = f_2(x) + \frac{1}{n}(x - x^2).$$

[30 marks]

) State Bohman-Korovkin theorem and use it to prove that there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of
olynomials such that $\lim_{n\to\infty} P_n = f$ for every $f \in C[0,1]$. Name the result. [25 marks]

3.

et

$$(S_nf)(x) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\,dt$$

nd define the Cesáro means (Fejer Operator) as

$$G_nf := \frac{1}{n}[S_0f + S_1f + \cdots S_{n-1}f].$$

a) Prove that

$$(G_nf)(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(t+x)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{t}{2}}\right)^2 dt$$

35 marks]

b) Show that the Fejer operators G_n are monetone linear [20 marks]

c) Show that $G_n 1 \to 1$, $(G_n\cos)(x) \to \cos x$, and $(G_n\sin)(x) \to \sin x$ as $n \to \infty$ and hence show that the Fejer operators of the Fourier series of a continuous 2π-periodic function converge uniformly to the function [45 marks]

(Hint: Use the trigonometric analogue of Korovkin's theorem that is given as follows: Let $\{L_n\}$ denote a sequence of monotone linear operators on $C_{2\pi}$. In order that $L_nf \to f$ (uniformly) for all $f \in C_{2\pi}$, it is necessary and sufficient that such convergence occurs for $f = 1$, cos, and sin.)

Q4.

(a) Define a Chebyshev system on a closed interval $[a,b]$ and show that $\{e^{a_1x}, e^{a_2x}, \ldots, e^{a_nx}\}$ is a Chebyshev system, where $a_1, a_2, \ldots, a_n$ are distinct real numbers. [25 marks]

(b) Chebyshev polynomial of degree n is defined as

$$T_n(x) := \cos(n\arccos x), \quad x \in [-1,1], \quad n = 0,1,2,\ldots$$

(i) Show that T_n satisfy the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2,3,\ldots$$

[25 marks]

(ii) Use (i) above and induction hypothesis to show that T_n is an algebraic polynomial of degree n with leading coefficient 2^{n-1}, $n \geq 1$ [25 marks]

(iii) Show that T_n satisfies the following differetial equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0, \quad \text{where } y = T_n(x).$$

[25 marks]

Q5.

(*a*) If g is a trigonometric polynomial of degree $\leq n$ with only cosine terms, show that there exists an ordinary polynomial $f(x)$ of degree $\leq n$ such that

$$g(\theta) = \cos\theta.$$

[20 marks]

(*b*) Let $f \in C[-1,1]$ and assume that for some $k \geq 1$, $f^{(k)} \in C[-1,1]$ and $g(\theta) = f(\cos\theta),\ \theta \in [0, 2\pi]$. Show the following:

(*i*)

$$\xi_n[g] = E_n[f], \quad n \geq 1;$$

[30 marks]

(*ii*)

$$E_n[f] \leq \frac{3}{2}\|f'\|_{[-1,1]}\frac{\pi}{n+1};$$

[25 marks]

(*iii*)

$$\xi_n[g] \leq \left(\frac{3}{2}\right)^{k+1}\pi^k \frac{\omega\left(f^{(k)}; \frac{\pi}{n-k+1}\right)}{(n+1)n(n-1)\ldots(n-k+2)};$$

where $E_n[f]$, $\xi_n[f]$ denote the errors in approximation of f by an ordinary and trigonometric polynomial of degree $\leq n$ respectively and $\omega(f;\bullet)$ denotes the modulus of continuity. [25 marks]

(You may assume that $E_n[f] = E_n[f-P]$, for any ordinary polynomial of degree $\leq n$)

Q6.

(*a*) In each of the following three inequalities, give the missing functions (in n) or (in n and x) that should appear in the following inequalities. Also name the inequalities. [30 marks]

(*i*) For trigonometric polynomials of degree n

$$\|R'\|_{[0,2\pi]} \leq \|R\|_{[0,2\pi]}$$

(*ii*) For ordinary (algebraic) polynomials P of degree n

$$\|P'\|_{[-1,1]} \leq \|P\|_{[-1,1]}$$

(*iii*) For ordinary (algebraic) polynomials P of degree n, and $x \in (-1,1)$

$$|P'(x)| \leq \|P\|_{[-1,1]}$$

Give two trigonometric polynomials of degree n for which there equality in (i) and one algebraic polynomial of degree n for which there is equality in (ii) above. [15 marks]

(b) Let $f \in C_{2\pi}$ and $0 < \alpha < 1$. Prove that the following are equivalent:

(I) There exists $A > 0$ such that $\xi_n[f] \leq An^{-\alpha}, \quad n \geq 1.$

(II) There exists $B > 0$ such that $|f(x) - f(y)| \leq B|x - y|^{\alpha}, \quad$ for $x, y \in [0, 2\pi]$.

[30 marks]

(c) Let $f(\theta) := |\theta|$, $\theta \in [-\pi, \pi]$ and extend f to R by 2π periodicity. Prove that there exists $A > 0$ such that

$$\xi_n[f] \leq An^{-1}, \quad n \geq 1.$$

but that if $\alpha > 1$, there does not exist $A > 1$ such that

$$\xi_n[f] \geq An^{-\alpha}, \quad n \geq 1.$$

[25 marks]

(Here $\xi_n[f]$ denotes the error in approximation of f by a trigonometric polynomials of degree n and R denotes the set of real numbers)