EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE-2008/2009

FIRST SEMESTER (FEB., 2010)

MT 304 - GENERAL TOPOLOGY

nswer all questions Time: Two hours

1. Define the following on a non-empty set X:
   - topology on X;
   - limit point of a subset of X;
   - dense subset of X.

(a) Let τ be the class of subsets of $\mathbb{N}$, set of all natural numbers, consisting of ϕ and all subsets of $\mathbb{N}$ of the form $E_n = \{n, n+1, n+2, \cdots\}$, where $n \in \mathbb{N}$. Show that τ is a topology on $\mathbb{N}$. Write down the open sets containing the element $5 \in \mathbb{N}$.

Determine the following:

i. limit points of the set $A = \{4, 13, 28, 37\}$;

ii. subsets E of $\mathbb{N}$ for which $E' = \mathbb{N}$, where E' is the set of all limit points of E;

iii. closed subsets of $(\mathbb{N}, \tau)$;

iv. closure of the sets $\{7, 24, 47, 85\}$ and $\{3, 6, 9, 12, \cdots\}$;

v. dense subsets of $\mathbb{N}$.

(b) Prove that a subset A of a topological space X is closed if and only if A contains each of its limit points.

Hence show that, $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ is not closed under the usual topology on $\mathbb{R}$, set of real numbers.

2. Define the terms *base* and *subbase* for a topological space.

(a) Let $\mathbb{B}$ be a class of subsets of a non-empty set X. Then prove that $\mathbb{B}$ is a base for some topology on X if and only if it satisfies the following properties:

(i) $X = \bigcup_{B \in \mathbb{B}} B$,

(ii) for any $B, B' \in \mathbb{B}$, if $p \in B \cap B'$ then there exists $B_p \in \mathbb{B}$ such that $p \in B_p \subseteq B \cap B'$.

(b) i. Let D be the discrete topology on $Y = \{a, b, c, d, e\}$. Find a subbase $\mathbb{S}$ for D, which does not contain any singleton sets.

ii. Let $\mathbb{S}$ be a subbase for a topology τ on X and A be a subset of X. Show that the class $\mathbb{S}_A = \{A \cap S : S \in \mathbb{S}\}$ is a subbase for the relative topology τ_A on A.

3. (a) What is meant by a function f from a topological space X to a topological space Y is *continuous at a point* $x_0 \in X$?

Let f be a function from a topological space (X, τ_1) into a topological space (Y, τ_2). Prove the following:

i. f is continuous if and only if $f^{-1}(G)$ is open in X for each open set G in Y.

ii. if S is a subbase for τ_2, then f is continuous if and only if $f^{-1}(A) \in \tau_1, \ \forall \ A \in S$.

(b) Define *Frechet space* (T_1) and *Hausdorff space* (T_2).

i. Prove that every T_2 space is T_1. Is the converse true? Justify your answer.

ii. Prove that a topological space (X, τ) is a T_1 space if and only if every singleton subset of X is closed.

4. Prove or disprove the following statements:

(a) continuous image of a compact set in a topological space is compact.

(b) in the usual topology on $\mathbb{R}$, the subset $(0, 1)$ is compact.

(c) the class of open intervals $A_n = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ satisfies the finite intersection property and $\bigcap_{n \in \mathbb{N}} A_n = \phi$.

(d) (X, τ) is a compact topological space if and only if for every class $\{F_i\}$ of closed subset of X, $\bigcap_i F_i = \phi$ implies $\{F_i\}$ contains a finite subclass $\{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$ with $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m} = \phi$.