Eastern University, Sri Lanka

First Examination in Science (2001/2002)

First Semester (April 2002)

MT 101 - Foundation of Mathematics

Answer all questions **Time allowed: 3 hours**

Q1.
Define the terms *tautology* and *contradiction* as applied to a logical proposition.

Explain what is meant by the statement that *two propositions are logically equivalent.*

(*a*) Let p, q and r be three propositions. Prove the following:

(*i*) $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$
(*ii*) $q \rightarrow (r \rightarrow p) \equiv (q \wedge (> p)) \rightarrow \; > r$
(*iii*) $[> p \wedge (> q \wedge r)] \vee (q \vee r) \vee (p \wedge r) \equiv \; > r$

(*b*) Test the validity of the following argument:
A person who drinks more alcohol sleeps more;
One who sleeps more commits less sin;
A person who commits less sin goes to heaven after death;
....
Therefore, a person who drinks more alcohol goes to heaven after death.

Q2.
Define the *power set of a set.*

Let A, B denote subsets of the set S.

(*a*) Write down a simple form of the negation of "$\forall x \in A,\ x \in B^c$".
Hence prove that

$$A \cap B = \phi \text{ if and only if } A \subseteq B^c, \text{ for } A, B \subseteq S.$$

Prove further that

$$\exists\, X \in \mathrm{P}(S) \text{ such that } (A \cap X) \cup (B \cap X^c) = \phi \text{ if and only if } A \cap B = \phi.$$

(Here $\mathrm{P}(S)$ denotes the power set of S)

(*b*) Prove that $A \cap B$ is the largest subset of S contained in both A and B.
Prove also that

$$\mathrm{P}(A) \cup \mathrm{P}(B) \subseteq \mathrm{P}(A \cup B) \text{ and } \mathrm{P}(A) \cap \mathrm{P}(B) = \mathrm{P}(A \cap B).$$

Give an example of a pair of sets C and D such that $\mathrm{P}(C) \cup \mathrm{P}(D) \neq \mathrm{P}(C \cup D)$.

(c) Let A, B, C and D be subsets of X and let
$P = (A \cap B) \cap (C \cup D)$, $Q = (A \cap D) \cup (B \cap C)$.
Prove that $P \subseteq Q$. Prove also that

$$P = Q \text{ if and only if } B \cap C \subseteq A \text{ and } A \cap D \subseteq B.$$

Q3.
Define the following:

(*i*) An *equivalence relation* on a set;

(*ii*) An *equivalence class* of an element in a set A.

(*a*) A relation R is defined on $\mathbf{N}$, the set of natural numbers, by

$$xRy \Leftrightarrow \exists\, n \in \mathbf{Z} \text{ such that } x = 2^n y \ (x, y \in \mathbf{N})$$

where $\mathbf{Z}$ denotes the set of all integers. Prove that R is an equivalence relation.

(*b*) Let R_1 and R_2 be two equivalence relations on a set X. Prove that $R_1 \cap R_2$ is an equivalence relation on X. Is $R_1 \cup R_2$ an equivalence relation on X? Justify your answer.

(*c*) Let A be any set and let $\sim$ be an equivalence relation on A. Prove the following:
$\forall\, a, b \in A$

(*i*) $[a] \neq \phi$
(*ii*) $a \sim b \Leftrightarrow [a] = [b]$
(*iii*) $b \in [a] \Leftrightarrow [a] = [b]$
(*iv*) Either $[a] = [b]$ or $[a] \cap [b] = \phi$
(Here $[x]$ denotes the equivalence class of $x \in A$)

Q4.
Define the following:

(*i*) Injective mapping;

(*ii*) Surjective mapping;

(*iii*) Bijective mapping;

(*vi*) Partially ordered set;

(*v*) First element of a partially ordered set;

(*vi*) Last element of a partially ordered set.

(a) Let $f : S \to T$ be a mapping. Prove that

(i) f is injective if and only if $f(A) \cap f(S \setminus A) = \phi, \ \forall A \subseteq S$.

(ii) f is injective then $f(A \cap B) = f(A) \cap f(B)$ for $A, B \subseteq S$.

(b) Let $f : A \to B$ and $g : B \to A$ be two mappings such that $g \circ f = I_A$ and $f \circ g = I_B$. Prove that f is bijective and $g = f^{-1}$.
(Here $\circ$ denotes the composition and f^{-1} denotes the inverse mapping of f)

(c) Show that a partially ordered set can have at most one first element and one last element.

Q5.
Define the following:

(i) A *divisor* of an integer;

(ii) Greatest common divisor (gcd) of two integers a and b;

(iii) Relatively prime;

(iv) The greatest integer of a real number.

(a) Let a, b, x, y and d be integers. Prove that $d/(ax + by)$ if d/a and d/b. Prove also that $8/(u^2 - v^2)$ for two odd integers u and v.

Explain whether it is possible to have 100 coins made of c cents, d dimes and q quarters, be worth exactly \$4.99. (Here 1 dime = 10 cents, 1 quarter = 25 cents)

(b) If $\gcd(a, b) = d$ then there exist integers x and y such that $ax + by = d$. Use this result to prove the following:

(i) If a, b and c are integers and c/ab then c/b, where a and c are relatively prime.

(ii) If a and b are integers, p/ab and $p \not| \ a$ then p/b, where p is a prime.

(c) State *Euclid's Divison Lemma.*

Find the greatest common divisor of $2m + 1$ and $2m - 1$ using the repeated application of Euclid's Lemma.

Q6.

(*a*) Define the *least common multiple* of two integers and prove that

$$\operatorname{lcm}(a,b)\gcd(a,b) = ab$$

where $\operatorname{lcm}(a,b)$ denotes the least common multiple of a and b and $\gcd(a,b)$ denotes the greatest common divisor of a and b.

(*b*) If $c \neq 0$, we say that $a \equiv b \pmod{c}$ if and only if $c/(a-b)$. Use this definition to show that

(*i*) if $ac \equiv bc \pmod{m}$ and $\gcd(c,m) = 1$ then $a \equiv b \pmod{\frac{m}{c}}$

(*ii*) if $a \equiv b \pmod{m_1}$ and $a \equiv b \pmod{m_2}$ then $a \equiv b(\operatorname{mod}(m_1 m_2))$.

What values of x will satisfy $2x \equiv 1 \pmod{7}$?

(*c*) Define the *Fibnacci Sequence* f_n and show that

(*i*) $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for $n \geq 1$

(*ii*) $f_n > \alpha^{n-2}$ for $n \geq 3$, where $\alpha = \frac{1+\sqrt{5}}{2}$.