EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 2002/2003

(June./July.'2003)

FIRST SEMESTER

MT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all questions

Time : Three hours

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ prove the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence prove that

$$(\underline{a} \wedge \underline{b}) \cdot [(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})] = [\underline{a} \cdot (\underline{b} \wedge \underline{c})]^2.$$

(b) If the vector $\underline{x}$ is given by the equation $\lambda \underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}$, $\underline{b}$ are constant vectors and λ is a non-zero scalar, show that

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

(c) Prove that if ϕ is a scalar field and $\underline{A}$ is a vector field then

$$\underline{\nabla} \wedge (\phi \underline{A}) = \phi(\underline{\nabla} \wedge \underline{A}) + \underline{\nabla}\phi \wedge \underline{A}.$$

Let $\underline{a}$ be a non-zero constant vector and $\underline{r}$ be a position vector of a point. If $r = |\underline{r}|$, find $\underline{\nabla}\left(\frac{\underline{a} \cdot \underline{r}}{r^5}\right)$.

Hence show that

$$\underline{\nabla} \wedge \left[\left(\frac{\underline{a} \cdot \underline{r}}{r^5}\right)\underline{r}\right] = \frac{\underline{a} \wedge \underline{r}}{r^5}.$$

2. (a) Define the following terms:

   i. A conservative vector field,

   ii. The scalar potential.

(b) If the force field $\underline{F} = \underline{\nabla}\phi$, where ϕ is single valued and has continuous partial derivative, show that the work done by moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ to another point $P_2 \equiv (x_2, y_2, z_2)$ in this field is independent of its path joining the two points.

Conversely, if $\int_C \underline{F} \cdot d\underline{r}$ is independent of path C joining any two points, show that there exists a function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

(c) Show that the field

$$\underline{F} = (2x\cos y + z\sin y)\underline{i} + (xz\cos y - x^2\sin y)\underline{j} + x\sin y\,\underline{k}$$

is conservative. Find the scalar potential ϕ such that $\underline{F} = \underline{\nabla}\phi$. Hence find the work done in moving an object in this field from $(-2, 0, 1)$ to $\left(3, \frac{\pi}{2}, 4\right)$.

3. (a) State the Divergence Theorem.

Verify the Divergence Theorem for

$$\underline{A} = (2x - z)\underline{i} + x^2y\underline{j} - xz^2\underline{k}$$

taken over the region bounded by $x = 0,\ x = 1,\ y = 0, y = 1,$ $z = 0, z = 1$.

(b) State Stoke's Theorem.

Verify Stoke's Theorem for $\underline{A} = xz\underline{i} - y\underline{j} + x^2y\underline{k}$ over the surface $2x + y + 2z = 8$ lying in the first octant.

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

respectively.

(a) A particle moves in a plane such that the radial and transverse components of velocity are λr and $\mu\theta$ respectively. Show that the radial and transverse components of the acceleration are $\left(\lambda^2 r - \frac{\mu^2\theta^2}{r}\right)$ and $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$ respectively.

(b) A particle of mass m rest on a smooth horizontal table attached to a fixed point on the table by a light elastic string of modulus mg and unstretched length $'a'$. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with the velocity $\sqrt{\frac{4ga}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t, then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g}{a}(r - a).$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial value.

5. State the angular momentum principle for motion of a particle.

A particle is projected horizontally along the inner surface of a smooth cone whose axis is vertical and vertex upwards. Find the reaction at any point in terms of the depth below the vertex. Show that the particle will leave the cone at a depth

$$\left[\frac{V^2h^2}{g\tan^2\alpha}\right]^{\frac{1}{3}},$$

below the vertex, where h is the initial depth, V is the initial velocity and α is the semi angle of the cone.

6. Establish the equation

$$\underline{F}(t) = m(t)\frac{d\underline{v}}{dt} + \underline{v}_0\frac{dm(t)}{dt}$$

for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{v}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate with a velocity $\underline{v}_0$ relative to the rocket.

A rocket, whose mass at time t is given by $m_0(1-\alpha t)$, where m_0 and α are constants, travels vertically upwards from rest at $t=0$. The ejected matter has a constant speed $\dfrac{4g}{\alpha}$ relative to the rocket. Assuming that the resistance of the atmosphere is $2m_0v\alpha$, where v is the speed of the rocket at time t, and the gravitational field g is constant. Show that

$$v = 3gt - g\alpha t^2.$$

Hence show that half of the original mass is left when the rocket reaches a height $\dfrac{g}{3\alpha^2}$.