EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - (2002/2003)

(JUNE/JULY, 2003)

FIRST SEMESTER

REPEAT

MT 203 - EIGENSPACE AND QUADRATIC FORMS

Answer all questions

Time: Two hours

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1. Define the term "eigenvalue" of a linear transformation. [10 marks]

   (a) Prove that an $n \times n$ square matrix A is similar to a diagonal matrix D whose diagonal elements are the eigenvalues of A if and only if A has n linearly independent eigenvectors. [25 marks]

   (b) Let A be a matrix of order n such that $A^2 = I$. Show that every eigenvalues of A is either 1 or -1. [25 marks]

   Let

$$A = \begin{pmatrix} 4 & 4 & 4 \\ 6 & 6 & 5 \\ -6 & -6 & -5 \end{pmatrix}.$$

   Find a non-singular matrix P such that $P^{-1}AP$ is diagonal.

   Hence find a matrix B such that $B^2 = A$. [40 marks]

2. (a) Define the terms "minimum polynomial" and "irreducible polynomial" of a square matrix. [20 marks]

   (b) State the Cayley-Hamilton theorem.

   By evaluating the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 5 & -1 \end{pmatrix},$$

show that $A^{-1} = -\frac{1}{12}(A^2 - 2A - 6I)$, where I is the identity matrix of order 3. [20 marks]

(c) Prove the following:

i. The characteristic polynomial of an $n \times n$ matrix A always divides the n^{th} power of its minimum polynomial.

ii. The characteristic polynomial and the minimum polynomial of an $n \times n$ matrix A have the same irreducible factors.

[40 marks]

(d) Let A and B be two arbitrary matrices in $F_{n \times n}$. Let M be the $(2n) \times (2n)$ matrix of the form $\begin{bmatrix} tI & A \\ B & I \end{bmatrix}$. By premultiplying M by a matrix $\begin{bmatrix} I & -A \\ O & I \end{bmatrix}$, prove that $\det M = \chi_{AB}(t)$, where $\chi_{AB}(t)$ is the characteristic polynomial of AB.

By postmultiplying M by a suitable matrix of the form $\begin{bmatrix} I & X \\ O & Y \end{bmatrix}$, deduce that AB and BA have the same eigenvalues, where I and O denote the identity matrix and zero matrix of order n respectively.

[20 marks]

3. Let λ_1 and λ_2 be two distinct roots of the equation $| A - \lambda B | = 0$, where A and B are real symmetric matrices and let u_1 and u_2 be two vectors satisfying $(A - \lambda_i B)u_i = 0$ for $i = 1, 2$.

Prove that $u_1{}^T B u_2 = 0$.

[30 marks]

Simultaneously reduce the following pair of quadratic forms.

$$\phi_1 = 9{x_1}^2 + 6{x_2}^2 + 8{x_3}^2 + 4x_2x_3 + 4x_3x_1 - 4x_1x_2;$$

$$\phi_2 = 5{x_1}^2 + 5{x_2}^2 + 12x_2x_3 - 12x_3x_1 + 8x_1x_2.$$

[70 marks]

4. What is meant by an "inner product" on a vector space?

Verify that the function $< .,. >$, defined by

$$< u, v > \; = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$$

is an inner product on $\mathbb{R}^2$, where $u = (x_1, x_2), v = (y_1, y_2)$.

[25 marks]

(a) If X is a finite dimensional inner product space and f is a linear functional on X, prove that there exists a unique vector $y \in X$ such that $f(x) = < x, y >, \;\; \forall x \in X$.

[25 marks]

(b) Let X be an inner product space and M be a finite dimensional subspace of X. Prove that $X = M \oplus M^{\perp}$, where $M^{\perp}$ is orthogonal complement of M and $\oplus$ denotes the direct sum.

[25 marks]

(c) State Gram-Schmidt process and use it to find the orthonormal set for span of S in $\mathbb{R}^4$,

where $S = \{(1, 0, -1, 0)^T, (0, 1, 2, 1)^T, (2, 1, 1, 0)^T\}$.

[25 marks]